The Potential of a Lens, and Allied Physical Problems.

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As a sequel to the article in this Journal, Vol. XXXIII, p. 311, on the "Potential of a Spherical Segment (a Lens)", some further applications may be cited, by which the solution can be given of allied problems in Electricity, Attraction, and Hydrodynamics.

1. Starting with U, the potential function (P. F.) of a spherical bowl of radius c cm, and uniform superficial density, σ, in g/cm^2 super, it was shown in the former article (cited as A. J. M.) that G denoting the gravitation constant, 666×10^{-10} in C. G. S. units,

$$\frac{U}{G\sigma} = c\Omega + r'\Omega', \tag{1}$$

(A. J. M., § 8), and this is Maxwell's P of § 670, "Electricity and Magnetism" (E. and M.); and then, as in A. J. M., § 13 (18),

$$\frac{1}{c}\frac{d}{dr}(Pr) = \frac{d}{dr}(r\Omega + c\Omega') = \Omega + r\frac{d\Omega}{dr} + c\frac{d\Omega'}{dr} = \Omega. \tag{2}$$

The P. F. U in (1) is composed of two terms, of which Ω in the first is the apparent area at P of the bowl or its base AB (fig. 1), and Ω is a P. F. satisfying Laplace's equation; and as U is a P. F. at P, it follows that the second term $r'\Omega'$ is also a P. F. at P, while Ω' is a P. F. at P', the inverse point of P in the spherical surface, Ω' representing the apparent area of the bowl or its base AB at P'.

Interpreted physically, Ω will represent the magnetic potential at P of a plate bounded by the circle AB and magnetized normally, or the equivalent electro-magnetic potential of a current round the rim; or it will represent the illumination on a page at P parallel to AB, due to skylight coming through the circle AB, as inside a chimney shaft or down a well, where the illumination of a sky of uniform brightness would be reduced by the fraction

$$\frac{\Omega}{2\pi} = 1 - \frac{h}{\sqrt{(h^2 + a^2)}},$$

at a depth h in the middle of a shaft or well of radius a.

2. The radial component F of the attraction along PC is given by

$$\frac{F}{G\sigma}=-\frac{d}{dr}(c\Omega+r'\Omega')=-c\frac{d\Omega}{dr}-r'\frac{d\Omega'}{dr}+\frac{c^2}{r^2}\Omega'=\frac{c^2}{r^2}\Omega'. \qquad (1)$$

Thus with P on the axis, and outside the convex side of the bowl at G, when P' is inside at O,

$$\Omega'=2\pi, \quad \frac{F}{G\sigma}=2\pi\frac{CA^2}{CG^2}=2\pi\cos^2\gamma. \qquad (2)$$

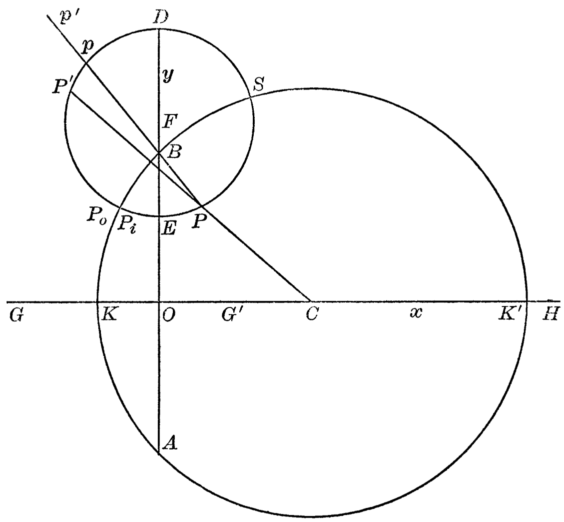

FIG. 1.

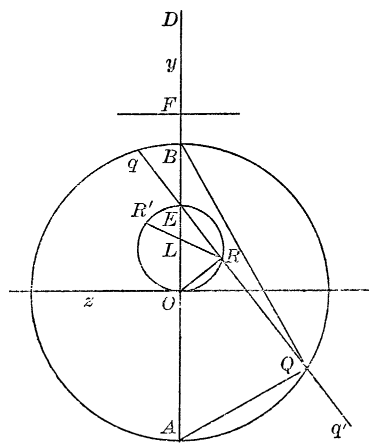

FIG. 2.

With P inside at O, and P' outside at G, Ω' is negative,

$$\Omega'=-4\pi+2\pi\frac{OG}{OA}=-4\pi+2\pi\sin\gamma,$$

$$\frac{F}{G\sigma}=2\pi(-2+\sin\gamma)\frac{CA^2}{CO^2}=2\pi\frac{-2+\sin\gamma}{\cos^2\gamma}. \qquad (3)$$

With P at K', $c=r=x'$, $\Omega=\Omega'$,

$$\frac{F}{G\sigma}=\Omega=2\pi\frac{OK'}{AK'}=2\pi\cos\tfrac{1}{2}\gamma; \qquad (4)$$

P outside, close to K, and P' inside, $r=c=r'$,

$$\frac{F}{G\sigma}=\Omega'=2\pi\frac{KO}{KA}=2\pi\sin\tfrac{1}{2}\gamma; \qquad (5)$$

P inside, and P' outside, close to K,

$$\frac{F}{G\sigma}=\Omega'=-4\pi+2\pi\sin\tfrac{1}{2}\gamma. \tag{6}$$

A particle at K, inside or outside, will stick to the surface of the bowl, in stable equilibrium; for if slightly displaced on a small smooth spot at K, it will beat time with a pendulum of length (A. J. M., § 7, p. 386),

$$\frac{g}{G\sigma\pi}\frac{KA^3}{OA^2}, \tag{7}$$

this length is $\dfrac{4}{3}\dfrac{\Delta}{\sigma}\dfrac{KA^3}{OA^2}$ of the earth's radius with a mean density Δ; or the beat is equal to $\sqrt{\left(\dfrac{1}{3}\dfrac{\Delta}{\sigma}\dfrac{KA^3}{OA^2}\right)}$ of the period of the grazing satellite.

3. In the previous demonstration it was assumed that the bowl was the segment of a sphere made by a plane; but as the result is independent of the size of the segment, it holds true when the segment is made small; and then by summation the result in (1) § 1 is seen to be unaltered in form when the bowl is bounded by any other curve.

This is evident by elementary geometry in fig. 3; the element dS of the spherical surface at E has the potential $\dfrac{dS}{EP}$ at P, and

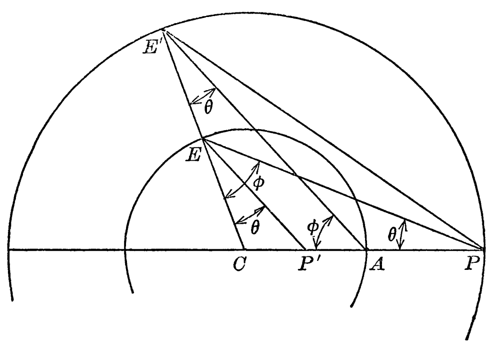

FIG. 3.

$$EP=CE\cos\phi+CP\cos\theta, \tag{1}$$

so that

$$\begin{aligned}\frac{dU}{G\sigma}=\frac{dS}{EP}&=\frac{dS}{EP^2}(CE\cos\phi+CP\cos\theta)\\ &=CE\frac{dS\cos\phi}{EP^2}+CP'\frac{dS\cos\theta}{EP'^2}\left(\text{because }\frac{CP}{CP'}=\frac{EP^2}{EP'^2}\right)\\ &=CE\cdot d\Omega+CP'\cdot d\Omega',\end{aligned} \tag{2}$$

reckoning Ω, Ω' positive when their aspect from P is the concave side of the bowl; and then by summation,

$$\frac{U}{G\sigma}=c\Omega+r'\Omega', \tag{3}$$

whatever the shape of the boundary rim of the bowl.

And for F, the radial force along PC of attraction at P,

$$\frac{dF}{G\sigma}=\frac{dS\cos\theta}{EP^2}=\frac{CP'}{CP}\frac{dS\cos\theta}{EP'^2}=\frac{r'}{r}d\Omega'=\frac{c^2}{r^2}d\Omega', \tag{4}$$

$$\frac{F}{G\sigma}=\frac{c^2}{r^2}\Omega'. \tag{5}$$

These two or three lines of geometry can thus replace some pages of analysis in Maxwell's E. and M.

For a complete sphere, and

$$P \text{ inside,}\quad \Omega=4\pi,\ \Omega'=0,\quad \frac{U}{G\sigma}=4\pi c,\quad F=0, \tag{6}$$

$$P \text{ outside,}\quad \Omega=0,\quad \Omega'=4\pi,\quad \frac{U}{G\sigma}=4\pi r'=\frac{4\pi c^2}{r},\quad \frac{F}{G\sigma}=\frac{4\pi c^2}{r^2}, \tag{7}$$

the well-known results for a spherical shell, and thence for a solid sphere given first by Newton in the "Principia," and of pioneering interest in justifying his theory of gravitation.

Because evidence has been found recently, by Prof. J. C. Adams, that Newton laid aside his calculations for nineteen years, till 1684, not only on account of his erroneous estimate of the size of the earth, at sixty land miles to the degree of latitude instead of sixty-nine; but also because Newton wanted to prove that the attraction of a spherical body like the earth on an external body, like an apple at the surface, was the same as if the earth was condensed into a particle at the centre, an assumption good enough for its attraction on a distant body like the moon, but requiring justification for an apple on the surface. As soon as this was clear, he set to work at once on the "Principia."

4. The theorem in § 1 is general, that if any P. F. at P is given as a function of r, $\mu=\cos\phi$, ψ, by $V=V(r,\mu,\psi)$, so that $V'=V\left(r'=\frac{c^2}{r},\mu,\psi\right)$ is a P. F. at the inverse point P', then

$$r'V(r',\mu,\psi)=\frac{c^2}{r}V\left(\frac{c^2}{r},\mu,\psi\right) \text{ is a P. F. at } P. \tag{1}$$

(W. D. Niven, L. M. S., VIII, 1876, p. 66; W. Burnside, L. M. S., XXV, 1893, p. 99).

The theorem is proved at once by Laplace's operator

$$r^2\frac{d^2}{dr^2}+2r\frac{d}{dr}+\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\psi^2},\tag{2}$$

which changes, for $r'=\frac{c^2}{r}$, into

$$r'^2\frac{d^2}{dr'^2}+\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\psi^2},\tag{3}$$

and this operator, acting as an annihilator on $r'V'$, reduces to Laplace's equation

$$r'^2\frac{d^2V'}{dr'^2}+2r'\frac{dV'}{dr'}+\frac{d}{d\mu}(1-\mu^2)\frac{dV'}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2V'}{d\psi^2}=0.\tag{4}$$

Or in the algebraical form employed by Niven, if V in $f(V, x, y, z)=0$ satisfies Laplace's operator $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0$, so also does the function

$$f\left(\frac{r}{c}V,\quad \frac{c^2}{r^2}x,\quad \frac{c^2}{r^2}y,\quad \frac{c^2}{r^2}z\right)=0.$$

A combination of V and $\frac{r'}{c}V'$, or $\frac{c}{r}V'$ will then give a P. F., where $V=V'$ on the sphere $r=c=r'$.

5. For a distribution symmetrical about the axis Ox, the P. F. V satisfies Laplace's equation in the form

$$\frac{d}{dx}\left(y\frac{dV}{dx}\right)+\frac{d}{dy}\left(y\frac{dV}{dy}\right)=0;\tag{1}$$

so that a function N can be assigned, the Stokes or stream function (S. F.), such that

$$y\frac{dV}{dx}=\frac{dN}{dy},\quad y\frac{dV}{dy}=-\frac{dN}{dx},\tag{2}$$

and the meridian curves of the surface of constants V and N are orthogonal.

The factor 2π of N in A. J. M., § 11, introduced by Maxwell has been omitted here, and the sign changed; so that here the slope or gradient of V is changed into the gradient of N by a rotation through a right angle against the clock or sun, *widdershins*. Also

$$\frac{dV}{dx}=\frac{1}{y}\frac{dN}{dy},\quad \frac{dV}{dy}=-\frac{1}{y}\frac{dN}{dx},\tag{3}$$

$$\frac{d^2V}{dxdy}=\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=-\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right),\tag{4}$$

$$\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=0.\tag{5}$$

Better then use the ordinary x, y coordinates instead of z, ϖ, or Maxwell's b, A; because at a large distance from the axis Ox, these equations (1) and (5) degenerate into

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0, \quad \frac{d^2N}{dx^2}+\frac{d^2N}{dy^2}=0, \tag{6}$$

as for plane orthogonal conjugate functions, V and N.

In polar coordinates in fig. 1, with

$$x = c\cos\gamma - r\cos\phi, \quad y=r\sin\phi, \tag{7}$$

$$\frac{d}{dr}=-\cos\phi\frac{d}{dr}+\sin\phi\frac{d}{dy}, \quad \frac{d}{rd\phi}=\sin\phi\frac{d}{dx}+\cos\phi\frac{d}{dy}, \tag{8}$$

$$\frac{dN}{dr}=-\cos\phi\frac{dN}{dx}+\sin\phi\frac{dN}{dy}=y\cos\phi\frac{dV}{dy}+y\sin\phi\frac{dV}{dx}$$
$$=y\frac{dV}{rd\phi}=\sin\phi\frac{dV}{d\phi}, \tag{9}$$

$$\frac{dN}{rd\phi}=\sin\phi\frac{dN}{dx}+\cos\phi\frac{dN}{dy}=-y\sin\phi\frac{dV}{dy}+y\cos\phi\frac{dV}{dr}$$
$$=-y\frac{dV}{dr}=-\sin\phi\frac{rdV}{dr}, \tag{10}$$

and the V gradient is changed into the N gradient by a rotation through a right angle with the clock and sun, *deasil.*

But here x and $r\cos\phi$ are antagonistic, so that V, *widdershins* with N with respect to x and y, becomes *deasil* in the polar coordinates r, ϕ.

With dn the outward normal element of an equipotential V, and ds the element of the meridian curve of V, normal element of N,

$$\frac{dN}{ds}=y\frac{dV}{dn}=4\pi\sigma y, \quad N_1-N_2=\int 4\pi\sigma y ds, \tag{11}$$

and this is twice the charge or induction through the zone S_1-S_2 on the equipotential V; σ denoting the electrical density of induction.

6. A relation similar to (1) § 4, holds for a Stokes stream **function** (S. F.) at P; denoting it by N, or $N(r, \mu=\cos\phi)$ and by $N'=N\left(r'=\frac{c^2}{r}, \mu\right)$ at P', then

$$\frac{r}{c}N' = \frac{c}{r'}N' \tag{1}$$

is a S. F. at P; and Laplace's operator in (2) § 4, is replaced for the S. F. by

$$r^2\frac{d^2}{dr^2}+(1-\mu^2)\frac{d^2}{d\mu^2}, \tag{2}$$

changing with $r=\dfrac{c^2}{r'}$ into

$$r'^2\frac{d^2}{dr'^2}+2r'\frac{d}{dr'}+(1-\mu^2)\frac{d^2}{d\mu^2}, \tag{3}$$

so that, operating as an annihilator on $\dfrac{cN'}{r'}$,

$$r'^2\frac{d^2N'}{dr'^2}+(1-\mu^2)\frac{d^2N'}{d\mu^2}=0. \tag{4}$$

Thus, if

$$y^2M(r,\mu)=r^2\sin^2\phi M(r,\cos\phi), \tag{5}$$

is a S. F. at P, then $y'^2M(r',\mu)$ is a S. F. at P'; and

$$\begin{aligned}\frac{r}{c}y'^2M\left(\frac{c^2}{r},\mu\right)&=\frac{r}{c}r'^2\sin^2\phi M\left(\frac{c^2}{r},\mu\right)\\&=\frac{c^3}{r}\sin^2\phi M\left(\frac{c^2}{r},\mu\right)=\frac{c^3}{r^3}y^2M\left(\frac{c^2}{r},\mu\right)\end{aligned} \tag{6}$$

is also a S. F. at P.

Axial differentiation $\dfrac{d}{dx}$ will give a new P. F. and S. F., such as $\dfrac{dV}{dx}$ and $\dfrac{dN}{dx}$; and since, in Maxwell's notation, with $A=y\cos\psi$,

$$\frac{dV}{dy}=\frac{dV}{dA}\cos\psi=\frac{1}{A}\frac{dN}{db}\cos\psi; \tag{7}$$

this transverse differentiation will give a new P. F., tesseral of the first order, suitable for use in a uniform field perpendicular to the axis Ox.

7. For a plane circular plate AB, not dished as a bowl, $c=\infty$, $\Omega'=0$, and the result in (1) § 1, changes, in Maxwell's coordinates A, b in E. and M. § 701, to the expression in A. J. M., § 3, p. 378.

$$\frac{W}{G\sigma}=aP-AQ-b\Omega, \tag{1}$$

$$\frac{1}{G\sigma}\left(\frac{dW}{da},\ \frac{dW}{dA},\ \frac{dW}{db}\right)=P,\ -Q,\ -\Omega; \tag{2}$$

illustrating the application of the complete Elliptic Integral, First, Second and Third, in P, Q, Ω; as P here represents the potential of the rim of the plate, Q and Ω the magnetic potential for uniform magnetization, in the plate and normal to it.

For any other boundary of a plane plate the normal component of the attraction, or the magnetic potential of uniform normal magnetization, is still given at a point P by the conical angle Ω of the plate as seen from P.

For if $d\alpha$ denotes a small element of area round a point Q on the plate, the normal component of the attraction of the element is

$$\frac{G\sigma d\alpha}{PQ^2}\cos\text{(angle between } PQ \text{ and normal)} = G\sigma d\Omega \tag{3}$$

and so in $G\sigma\Omega$ for the whole area.

Thus, for an infinite plate, $\Omega=2\pi$, and the field of the attraction is uniform and $2\pi G\sigma$; changing to $-2\pi G\sigma$ in crossing the plate, a total change of $4\pi G\sigma$, in accordance with a general theorem.

The S. F. L of the plate, with the sign changed to that in A. J. M., § 12, p. 391, and omitting 2π, is then given by

$$\frac{L}{G\sigma} = \tfrac{1}{2}abP + \tfrac{1}{2}bAQ + \tfrac{1}{2}(a^2-A^2)\Omega, \tag{4}$$

$$\frac{dL}{dA} = A\frac{dU}{db} = -G\sigma A\Omega, \quad \frac{dL}{db} = -A\frac{dU}{dA} = G\sigma AQ, \tag{5}$$

as in § 3; and the S. F. of P, the rim P. F., is $bP+a\Omega$.

8. Here, as in A. J. M., § 20, p. 405, for the flat circular plate, with $PQ=r$, $\theta=2\omega$, $r^2=r_1^2\cos^2\omega+r_2^2\sin^2\omega$,

$$Q=\int_0^{2\pi}\frac{-a\cos\theta d\theta}{r} = \frac{8a}{r_1+r_2}\,\frac{K-E(\varkappa)}{\varkappa}, \quad P-Q=\int\frac{a(1+\cos\theta)d\theta}{r}$$

$$=\frac{8a}{r_1}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\omega d\omega}{\Delta\omega} = \frac{8a}{r_1}\,\frac{E(\gamma)-\gamma'^2G}{\gamma^2} = \frac{8a}{r_1+r_2}\,\frac{E(\varkappa)-(1-\varkappa)K}{\varkappa}, \tag{1}$$

$$P=\int_0^{2\pi}\frac{ad\theta}{r} = \int_0^{\frac{1}{2}\pi}\frac{4ad\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}} = \int_{r_2}^{r_1}\frac{4adr}{\sqrt{(r_1^2-r^2\cdot r^2-r_2^2)}}$$

$$=\frac{4aG}{r_1} = \frac{8aK}{r_1+r_2}\to\frac{2\pi a}{R}, \tag{2}$$

where G, K is the complete quarter period to comodulus $\gamma'=\dfrac{r_2}{r_1}$, or modulus $\varkappa=\dfrac{r_1-r_2}{r_1+r_2}$, and R is Gauss's arithmetic-geometric mean (A. G. M) of r_1 and r_2.

Inserting some further intermediate values of the series of quadric transformations, such as

$$\lambda=\frac{1-\varkappa'}{1+\varkappa'}=\left(\frac{\sqrt{r_1}-\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}}\right)^2,\quad K=(1+\lambda)L=\frac{r_1+r_2}{\frac{1}{2}(\sqrt{r_1}+\sqrt{r_2})^2}L;$$

$$\mu=\frac{1-\lambda'}{1+\lambda'}=\left(\frac{\sqrt{\frac{r_1+r_2}{2}}-\sqrt[4]{r_1r_2}}{\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}}\right)^2,\quad L=(1+\mu)M=\frac{(\sqrt{r_1}+\sqrt{r_2})^2}{\frac{1}{2}\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}M;$$

$$\nu=\frac{1-\mu'}{1+\mu'}=\left(\frac{\frac{\sqrt{r_1}+\sqrt{r_2}}{2}-\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}}{\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}}\right)^2,$$

$$M=(1+\nu)N=\frac{\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}{\frac{1}{2}\left(\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}\right)^2}N; \tag{3}$$

the ring potential of a mass m round the circle AB is given by

$$\frac{m}{r_1}\frac{G}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{2}(r_1+r_2)}\frac{K}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{4}(\sqrt{r_1}+\sqrt{r_2})^2}\frac{L}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{4}\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}\frac{M}{\frac{1}{2}\pi}$$

$$=\frac{m}{\frac{1}{4}\left(\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}\right)^2}\frac{N}{\frac{1}{2}\pi}\rightarrow\frac{m}{R} \tag{4}$$

For a point close to the rim, r_2 is small, and r_1, $2A$ may be replaced by $2a$; P is then large, and Q too, but $P-Q$ is finite; and as in A. J. M., § 22, writing it

$$P=\int_0^{\frac{1}{2}\pi}\frac{4a\,d\omega}{r}=\int\frac{4a\sin\omega\,d\omega}{r}+\int\frac{4a(1-\sin\omega)\,d\omega}{r}, \tag{5}$$

the first integral

$$\int_\omega^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{r}=\int_{r_2}^{r}\frac{4a\sin\omega\,dr}{\sqrt{(r_1^2-r^2\cdot r^2-r_2^2)}}$$

$$=\frac{4a}{\sqrt{(r_1^2-r_2^2)}}\int\frac{dr}{\sqrt{(r^2-r_2^2)}}=\frac{4a}{\sqrt{(r_1^2-r_2^2)}}\operatorname{ch}^{-1}\frac{r}{r_2}, \tag{6}$$

$$\int_0^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{r}=2\sqrt{\frac{a}{A}}\operatorname{ch}^{-1}\frac{r_1}{r_2}=2\sqrt{\frac{a}{A}}\log\frac{r_1+\sqrt{(r_1^2-r_2^2)}}{r_2}\rightarrow 2\log\frac{2r_1}{r_2}$$

and the second integral, with $r>r_1\cos\omega$, $\Delta\omega>\cos\omega$,

$$\int_0^{\frac{1}{2}\pi}\frac{4a(1-\sin\omega)\,d\omega}{r}<\frac{4a}{r_1}\int\frac{1-\sin\omega}{\cos\omega}d\omega \text{ or } 2\int\frac{\cos\omega\,d\omega}{1+\sin\omega}=2\log 2, \tag{8}$$

so that we can take

$P=2 \log \frac{4r_1}{r_2}$ + small terms, making

$G \to \log \frac{4r_1}{r_2} = \log \frac{4}{\gamma'}$, practically, when γ' is small; and then

$G=2 \log 2\frac{1+\kappa}{\kappa'} = 2 \log \frac{4}{\kappa'} = 2K$; or $\gamma' = 4e^{-G} = 4e^{-\frac{1}{2}\pi \frac{G}{G'}}$, $\kappa' = 4e^{-\frac{1}{2}\pi \frac{K}{K'}}$,

as they may be written, with $G'=K'=\frac{1}{2}\pi$; (9)

$$P-Q=\frac{8a}{r_1}\int\frac{\cos^2\omega d\omega}{\Delta\omega} < \frac{8a}{r_1}\int \cos\omega d\omega, \text{ or } 4; \tag{10}$$

$$P+Q=2P-4+\text{ small terms } =4 \log \frac{4r_1}{er_2} + \text{ small terms}; \tag{11}$$

and this is large as r_2 is small.

Thus, for example, the capacity of a ring AB, of small circular cross section πc^2 may be taken, with $r_2=c$, $r_1=2a$,

$$\frac{2\pi a}{P} = \frac{\pi a}{\log \frac{8a}{c}} = \frac{\text{circumference of } AB}{\log \frac{64 \text{ area of circle } AB}{\text{area of cross section}}} \tag{12}$$

For instance, with $a=10c$, the capacity is $\frac{\pi a}{\log_e 80} = \frac{\pi a}{4.382} = 0.7168a$.

So also for the potential of a circular plate, in exact functions tabulated numerically, instead of in an approximation by series, as in Thomson and Tait, § 546.

9. When the bowl in fig. 1 is insulated and electrified, the electrical potential can be written, in analogy with the potential of the bowl itself in (1), § 1,

(W. Thomson, Liouville, Oct. 8, 1845, "Electrical Papers," XVIII, p. 178; J. C. Maxwell, "Scientific Papers," II, p. 303; Ferrers, Q. J. M., XVIII, 1881, p. 97; Gallop, Q. J. M., XXI, 1886, p. 229.)

$$V=\omega+\frac{c}{r}\omega'=\omega+\frac{r'}{c}\omega', \quad cV=c\omega+r'\omega', \tag{1}$$

where, in fig. 1, ω and ω' are plane angles, given by

$$\sin\omega=\frac{2a}{r_1+r_2}, \quad \sin\omega'=\frac{2a}{r_1'+r_2'}=\frac{2r\sin\gamma}{r_1+r_2}, \tag{2}$$

$$r_1=PA, \quad r_2=PB, \quad r_1'=P'A, \quad r_2'=P'B, \quad OP=r, \quad OP'=r'=\frac{c^2}{r}, \tag{3}$$

$$AB=2a, \quad ACB=2\gamma, \quad a=c\sin\gamma, \quad OC=c\cos\gamma, \quad OA=c, \tag{4}$$

$$\frac{r_1'}{r}=\frac{AP'}{AP}=\frac{DP'}{DP}=\frac{r'-c}{c-r}=\frac{r'}{c}=\frac{c}{r}=\frac{r_2'}{r_2}=\frac{r_1'+r_2'}{r_1+r_2}=\frac{\sin\omega}{\sin\omega'}, \tag{5}$$

the ratio of the line elements or relative magnification at P and P'; and $\sin\omega$, $\sin\omega'$ is the excentricity of the ellipse with foci at A, B, passing through P and P'.

The statement in (1) is verified, because ω and $\frac{c}{r}\omega'$ are P. F.'s at P; and over the bowl AKB, $r=r'=c$, $r_1'=r_1$, $r_2'=r_2$, $\sin\omega'=\sin\omega$, $\omega'=\pi-\omega$, so that $V=\pi$; while $\omega=\omega'$, $V=2\omega$ over the remaining part $AK'B$ of the spherical surface. At infinity, $r_1=r_2=r=\infty$, $r'=0$,

$$\omega=\sin\omega=\frac{a}{r}=\frac{c\sin\gamma}{r},\ \sin\omega'=\sin\gamma,\ V(\infty)=\frac{c\sin\gamma}{r}+\frac{c\gamma}{r}=\frac{E}{r}, \tag{6}$$

so that the charge $E=c(\gamma+\sin\gamma)$.

The term ω in V is the potential of the electrification of the flat circular disc AB, insulated and at potential $\frac{1}{2}\pi$; but the term $\frac{c}{r}\omega'$ is obtained by inversion of the disc AB with respect to C, as the potential of a bowl AHB on the base AB, part of a spherical surface passing through C, when this bowl is earthed and influenced by a point charge $-\frac{1}{2}\pi c$ at C; because $\omega'=\frac{1}{2}\pi$ over AHB.

The sum of the two terms is then the electric potential of the insulated bowl on the base AB, centre at C.

10. The difference of the two terms

$$V'=-\omega+\frac{c}{r}\omega' \tag{1}$$

is also a P. F., zero over the spherical bowl $AK'B$, where $r=c$, $\omega'-\omega=0$; but over AKB,

$$\omega'+\omega=\pi,\quad V=\pi-2\omega=2\omega'-\pi. \tag{2}$$

At infinity, $r_1=r_2=r=\infty$, $r'=0$;

$$\omega=\sin\omega=\frac{a}{\gamma}=\frac{c\sin\gamma}{r},\quad \sin\omega'=\sin\gamma,\quad V'=\frac{c}{r}(\gamma-\sin\gamma), \tag{3}$$

so that the charge is $c(\gamma-\sin\gamma)$.

At the centre C, where $r=0$, $\omega=\gamma$,

$$\frac{c}{r}\omega'=\sin\gamma,\quad V=-\gamma+\sin\gamma. \tag{4}$$

Mr. J. R. Wilton gives (*Messenger of Mathematics*, p. 96, August, 1914),

$$\phi=-\omega+\frac{c}{r}(\pi-\omega') \tag{5}$$

as the P. F. of the bowl AKB, uninsulated, in presence of a point charge πc at the centre C.

11. The S. F. A of the P. F. ω is then found to be given by

$$A=\tfrac{1}{2}\sqrt{[AB^2-(PA-PB)^2]}=\sqrt{(PA\cdot PB)}\sin\tfrac{1}{2}APB, \tag{1}$$

so that A is the semi-conjugate axis of the confocal hyperbola through P, and the meridian curves of constant ω and A are confocal ellipses and hyperbolas.

More generally, any oblate spheroid of which the disc AB is the focal circle, if insulated and electrified with a charge E, will have a P. F. V and S. F. N given at an external point P in the meridian plane APB by

$$V=\frac{2E}{AB}\sin^{-1}\frac{AB}{PA+PB},\quad N=E\sqrt{\left[1-\left(\frac{PA-PB}{AB}\right)^2\right]} \tag{2}$$

as this verifies at infinity, where $PA=PB=PO=$ infinity, and

$$V=\frac{2E}{AB}\sin^{-1}\frac{AB}{2PO}=\frac{2E}{AB}\cdot\frac{AB}{2PO}=\frac{E}{PO}. \tag{3}$$

The electrical density σ at a point Q on the spheroid will be given by

$$\sigma=\frac{E}{2\pi}\cdot\frac{1}{QA+QB}\cdot\frac{1}{\sqrt{(QA\cdot QB)}}=\frac{\text{electrical force}}{4\pi}, \tag{4}$$

$$\text{electrical force}=4\pi\sigma=\frac{E}{\tfrac{1}{2}(QA+QB)\sqrt{(QA\cdot QB)}}, \tag{5}$$

electrical charge on the zone $QK=$ half the difference of the S. F. at K and $Q=\tfrac{1}{2}E\left\{1-\sqrt{\left[1-\left(\frac{QA-QB}{AB}\right)^2\right]}\right\}$. (6)

12. Putting

$$\sqrt{\left[\left(\frac{r_1+r_2}{2}\right)^2-a^2\right]}=B,\ \text{with}\sqrt{\left[a^2-\left(\frac{r_1-r_2}{2}\right)^2\right]}=A, \tag{1}$$

as before in (1) § 11, where

$$r_1^2=x^2+(y+a)^2,\quad r_2^2=x^2+(y-a)^2,\quad r_1^2-r_2^2=4ay, \tag{2}$$

$$\frac{dr_1}{dx}=\frac{x}{r_1},\quad \frac{dr_2}{dx}=\frac{x}{r_2},\quad \frac{dr_1}{dy}=\frac{y+a}{r_1},\quad \frac{dr_2}{dy}=\frac{y-a}{r_2}, \tag{3}$$

$$B=\sqrt{(\tfrac{1}{2}\cdot r_1r_2+x^2+y^2-a^2)},\quad A=\sqrt{(\tfrac{1}{2}\cdot r_1r_2-x^2-y^2+a^2)}, \tag{4}$$

$$B^2+A^2=r_1r_2,\quad B^2-A^2=x^2+y^2-a^2,\quad AB=ax, \tag{5}$$

$$A,\ B=\tfrac{1}{2}\sqrt{(r_1r_2+2ax)}\mp\tfrac{1}{2}\sqrt{(r_1r_2-2ax)}, \tag{6}$$

$$\omega=\sin^{-1}\frac{2a}{r_1+r_2}=\cos^{-1}\frac{2B}{r_1+r_2}, \tag{7}$$

$$\frac{d\omega}{dx}=\frac{-\dfrac{2a}{r_1+r_2}\left(\dfrac{dr_1}{dx}+\dfrac{dr_2}{dx}\right)}{2B}=-\frac{ax}{Br_1r_2}=-\frac{A}{r_1r_2} \tag{8}$$

$$\frac{dA}{dy}=\frac{-\frac{1}{4}(r_1-r_2)\left(\frac{dr_1}{dy}-\frac{dr_2}{dy}\right)}{A}=\frac{-(r_1-r_2)\left(\frac{y+a}{r_1}-\frac{y-a}{r_2}\right)}{4A}$$

$$=\frac{y(r_1-r_2)^2-a(r_1^2-r_2^2)}{4Ar_1r_2}=\frac{y(a^2-A^2)-a^2y}{Ar_1r_2}=-\frac{yA}{r_1r_2}=y\frac{d\omega}{dx}, \quad (9)$$

$$\frac{d\omega}{dy}=\frac{\frac{-2a}{r_1+r_2}\left(\frac{y+a}{r_1}+\frac{y-a}{r_2}\right)}{2B}=\frac{-ay(r_1+r_2)+a^2(r_1-r_2)}{Br_1r_2(r_1+r_2)}$$

$$=\frac{-ay(r_1^2-r_2^2)+a^2(r_1-r_2)^2}{Br_1r_2(r_1^2-r_2^2)}=\frac{-a^2y^2+a^2(a^2-A^2)}{Br_1r_2ay}$$

$$=\frac{ax^2-aB^2}{Br_1r_2y}=\frac{xAB-aB^2}{Br_1r_2y}=\frac{xA-aB}{r_1r_2y}, \quad (10)$$

$$\frac{dA}{dx}=\frac{-\frac{1}{4}(r_1-r_2)\left(\frac{x}{r_1}-\frac{x}{r_2}\right)}{A}=\frac{x(r_1-r_2)^2}{4Ar_1r_2}$$

$$=\frac{x(-A^2+a^2)}{Ar_1r_2}=\frac{-xA^2+aAB}{Ar_1r_2}=\frac{-xA+aB}{r_1r_2}=-y\frac{d\omega}{dy} \quad (11)$$

which proves, as defined in (2), § 5, that A is the S. F. of the P. F. ω, as stated above in § 11.

Similarly, we prove that B is the S. F. of the P. F.

$$\omega_1=\text{ch}^{-1}\frac{2a}{r_1-r_2}=\text{sh}^{-1}\frac{2A}{r_1-r_2}, \quad (12)$$

and ω_1 can be the P. F. of the electrification of the infinite plate with the circular hole AB cut out, or of any confocal hyperboloid of revolution, the electric charge being infinite.

Then B is the semi-minor axis of the confocal ellipse through P, and

$$B=a\cot\omega=x\coth\omega_1=\sqrt{(r_1r_2)}\cos\tfrac{1}{2}APB, \quad (13)$$

$$A=x\tan\omega=a\ \text{th}\,\omega_1=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}APB; \quad (14)$$

these relations help to settle a doubtful sign.

13. The S. F. at P of the P. F. ω being

$$A=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}APB=\sqrt{\left[a^2-\left(\frac{r_1-r_2}{2}\right)^2\right]}, \quad (1)$$

as proved by the preceding differentiations, the S. F. at P' of the P. F. ω' is

$$A'=\sqrt{(r_1'r_2')}\sin\tfrac{1}{2}AP'B=\sqrt{\left[a^2-\left(\frac{r_1'-r_2'}{2}\right)^2\right]}$$

$$=\sqrt{\left[c^2\sin^2\gamma-\frac{c^2}{r^2}\left(\frac{r_1-r_2}{2}\right)^2\right]}=\frac{c}{r}\sqrt{\left[r^2\sin^2\gamma-\left(\frac{r_1-r_2}{2}\right)^2\right]}, \quad (2)$$

and then by (1), § 6,

$$C = \sqrt{(r_1 r_2)} \sin \tfrac{1}{2} AP'B = \sqrt{\left[r^2 \sin^2 \gamma - \left(\frac{r_1 - r_2}{2}\right)^2\right]} \tag{3}$$

is a S. F. at P.

But we must not suppose that it has the P. F. $\frac{c}{r}\omega'$; the S. F. of this P. F. $\frac{c}{r}\omega'$ must be determined to have the S.F. by addition of the whole P.F. $\omega + \frac{c}{r}\omega'$.

By analogy with the S. F. of the material of the bowl, given in A. J. M., § 12, where, with the sign changed, AQ is the S. F. of the P. F. Ω, satisfying the relations of A. J. M., § 13,

$$\frac{dAQ}{dA} = -A\frac{d\Omega}{db}, \quad \frac{dAQ}{db} = A\frac{d\Omega}{dA}, \tag{4}$$

and so there the P. F. $\frac{c}{r}\Omega'$ has the S. F. $aP - c\Omega \cos V - c\Omega' \cos \phi$; the analogy shows that the S. F. of the P. F. $\frac{c}{r}\omega'$ will contain a term $c\omega' \cos \phi$, and Mr. Wilton has found (*Messenger of Mathematics*, p. 70, August, 1914), that the complete expression is

$$c\omega - c\omega' \cos \phi, \text{ the S. F. of the P. F. } \frac{c}{r}\omega'. \tag{5}$$

Thus the P. F.

$$V = \omega + \frac{c}{r}\omega' \tag{6}$$

has the S. F.

$$N = A + c\omega - c\omega' \cos \phi. \tag{7}$$

This is verified by the differentiation

$$\begin{aligned}\frac{dN}{dx} + y\frac{dV}{dy} &= \frac{dA}{dx} + y\frac{d\omega}{dy} + c\frac{d\omega}{dx} - c\frac{d\omega'}{dx}\cos\phi + c\omega' \sin\phi \frac{\sin\phi}{r} \\ &\qquad + r\sin\phi\left(\frac{c}{r}\frac{d\omega'}{dy} - \frac{c}{r^2}\omega' \sin\phi\right) \\ &= c\frac{d\omega}{dx} + c\left(-\frac{d\omega'}{dx}\cos\phi + \frac{d\omega'}{dy}\sin\phi\right) = c\left(\frac{d\omega}{dx} + \frac{d\omega'}{dr}\right) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned}\frac{dN}{dy} - y\frac{dV}{dx} &= \frac{dA}{dy} - y\frac{d\omega}{dx} + c\frac{d\omega}{dy} - c\frac{d\omega'}{dy}\cos\phi + c\omega' \sin\phi \frac{\cos\phi}{r} \\ &\qquad - r\sin\phi\left(\frac{c}{r}\frac{d\omega'}{dx} + \frac{c}{r^2}\omega' \cos\phi\right) \\ &= c\frac{d\omega}{dy} - c\left(\frac{d\omega'}{dx}\sin\phi + \frac{d\omega'}{dy}\cos\phi\right) = c\left(\frac{d\omega}{dy} - \frac{d\omega'}{rd\phi}\right) = 0. \end{aligned} \tag{9}$$

These relations in (8) (9) are verified by putting

$$B'=\sqrt{\left[\left(\frac{r_1'+r_2'}{2}\right)^2-a^2\right]}=\frac{c}{r}D, \text{ with } A'=\sqrt{\left[a^2-\left(\frac{r_1'-r_2'}{2}\right)^2\right]}=\frac{c}{r}C, \quad (10)$$

as before; and then, differentiating,

$$\frac{d\omega'}{rd\phi}=\frac{r'}{r}\frac{d\omega'}{r'd\phi}=\frac{c^2}{r^2}\left(\frac{d\omega'}{dx'}\sin\phi+\frac{d\omega'}{dy'}\cos\phi\right)$$

$$=\frac{c^2}{r^2}\left(\frac{-A'\sin\phi}{r_1'r_2'}+\frac{x'A'-aB'}{r_1'r_2'y'}\cos\phi\right)=-\frac{A'\sin\phi}{r_1r_2}+\frac{x'A'-aB'}{r_1r_2\frac{c^2}{r^2}y}\cos\phi$$

$$=-\frac{\frac{c}{r}C\sin\phi}{r_1r_2}+\frac{\left(c\cos\gamma-\frac{c^2}{r}\cos\phi\right)\frac{c}{r}C\cos\phi-c\sin\gamma\frac{c}{r}D\cos\phi}{r_1r_2\frac{c^2}{r^2}y}$$

$$=\left[-\frac{c}{r}Cr\sin^2\phi+\left(c\cos\gamma-\frac{c^2}{r}\cos\phi\right)\frac{r}{c}C\cos\phi-c\sin\gamma\frac{r}{c}D\cos\phi\right]\div r_1r_2y$$

$$=[(-c+r\cos\phi\cos\gamma)C-Dr\sin\gamma\cos\phi]\div r_1r_2y$$

$$=[x(-C\cos\gamma+D\sin\gamma)-a(C\sin\gamma+D\cos\gamma)]\div r_1r_2y$$

$$=\frac{xA-aB}{r_1r_2y}=\frac{d\omega}{dy}; \quad (11)$$

because AE, BE bisect the angles PAP', PBP' if the spherical surface cuts CPP' in E; and (APC, AEC, $AP'C$) (BPC, BEC, $BP'C$) are angles in arithmetical progression, and so is their difference, so that

$$\tfrac{1}{2}APB+\tfrac{1}{2}AP'B=AEB=\gamma, \quad (12)$$

$$A=\surd(r_1r_2)\sin\tfrac{1}{2}APB=\surd(r_1r_2)\sin(\gamma-\tfrac{1}{2}AP'B)=D\sin\gamma-C\cos\gamma, \quad (13)$$

$$B=\surd(r_1r_2)\cos\tfrac{1}{2}APB=\surd(r_1r_2)\cos(\gamma-\tfrac{1}{2}APB)=D\cos\gamma+C\sin\gamma, \quad (14)$$

as if A, B and C, D were orthogonal components of the same vector.

Differentiating again

$$\frac{d\omega'}{dr}=-\frac{r'}{r}\frac{d\omega'}{dr'}=-\frac{c^2}{r^2}\left(-\frac{d\omega'}{dx'}\cos\phi+\frac{d\omega'}{dy'}\sin\phi\right)$$

$$=-\frac{c^2}{r^2}\left(\frac{A'\cos\phi}{r_1'r_2'}+\frac{x'A'-aB'}{r_1'r_2'y'}\sin\phi\right)$$

$$=-\frac{\frac{c}{r}C\cos\phi}{r_1r_2}-\frac{\left(c\cos\gamma-\frac{c^2}{r}\cos\phi\right)\frac{c}{r}C\sin\phi-c\sin\gamma\frac{c}{r}D\sin\phi}{r_1r_2\frac{c^2}{r}\sin\phi}$$

$$=\left[-\frac{c}{r}C\cos\phi-\left(\cos\gamma-\frac{c}{r}\cos\phi\right)C+D\sin\gamma\right]\div r_1r_2$$

$$=\frac{-C\cos\gamma+D\sin\gamma}{r_1r_2}=\frac{A}{r_1r_2}=-\frac{d\omega}{dx}. \quad (15)$$

Mr. Wilton has considered also the S. F.

$$a\frac{\cos\omega+\cos\omega'}{\sin\omega}=B+D, \tag{16}$$

which vanishes over the bowl, and has the P. F.

$$\omega\sin\gamma+\omega_1(1+\cos\gamma), \tag{17}$$

and investigated the physical interpretation. So also

$$a\frac{\text{sh}\,\omega_1+\text{sh}\,\omega_1'}{\text{ch}\,\omega_1}=A+C, \tag{18}$$

is a S. F. with a P. F.

$$\omega(1-\cos\gamma)+\omega_1\sin\gamma. \tag{19}$$

Other combinations can be made and interpreted (J. R. Wilton, *Messenger of Mathematics*, p. 75, August, 1914).

14. Over the bowl AKP_iB,

$$r=c,\quad r_1=2c\sin\tfrac{1}{2}(\gamma+\phi),\quad r_2=2c\sin\tfrac{1}{2}(\gamma-\phi),\quad r_1+r_2=4c\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\phi, \tag{1}$$

$$\sin\omega=\frac{2a}{r_1+r_2}=\frac{2c\sin\gamma}{4c\sin\frac{1}{2}\gamma\cos\frac{1}{2}\phi}=\frac{\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}\phi}=\frac{BK'}{PK'}. \tag{2}$$

But over $BK'A$, the rest of the spherical surface,

$$r_1=2c\sin\tfrac{1}{2}(\phi+\gamma),\quad r_2=2c\sin\tfrac{1}{2}(\phi-\gamma),\quad r_1+r_2=4c\cos\tfrac{1}{2}\gamma\sin\tfrac{1}{2}\phi \tag{3}$$

$$\sin\omega=\frac{\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\phi}=\frac{AK}{QK}. \tag{4}$$

The P. F. $V=\omega+\dfrac{c}{r}\omega'$ must be adjusted as a single valued function, although ω and ω' are multiple-valued.

Starting from a point S on the spherical surface $AK'B$, where

$$\omega=\omega'=\sin^{-1}\frac{AK}{SK}, \tag{5}$$

taken as the acute angle, and $V=2\omega$, let the point P travel from S to P_i on the interior of the bowl, and for simplicity along the orthogonal circle, with limiting points A and B.

Then P' will travel from S along this circle in the opposite direction, and will reach a point P_0 on the outside of the bowl, coincident with P_i. In this path P' does not cross the base AB of the bowl, and ω' never reaches $\frac{1}{2}\pi$, and $\cos\omega'$ remains positive.

But P in crossing the base AB makes $\omega=\frac{1}{2}\pi$; and beyond AB and up to P_i we must take $\omega>\frac{1}{2}\pi$, and $\cos\omega$ negative.

Thus if ω_0, ω_i denotes the value of ω according as it is reached from S by a path to the outside at P_0, or inside at P_i of the bowl,

$$\sin\omega_0 = \sin\omega_i = \frac{BK'}{PK'}, \quad \cos\omega_0 = -\cos\omega_i, \quad \omega_0+\omega_i = \pi, \tag{6}$$

making π the potential on the bowl; and we can put

$$\omega_0 = \tfrac{1}{2}(\pi-\theta), \quad \omega_i = \tfrac{1}{2}(\pi+\theta), \quad \cos\tfrac{1}{2}\theta = \frac{BK'}{PK'}, \tag{7}$$

where $\frac{1}{2}\theta$ is a positive acute angle. At S, the S. F. is

$$N(S) = \tfrac{1}{2}\sqrt{[AB^2-(SA-SB)^2]} + c(1-\cos\phi)\sin^{-1}\frac{AK}{SK}; \tag{8}$$

and travelling from S to P_i,

$$N(P_i) = A + c\omega_i - c\omega_0\cos\phi = \tfrac{1}{2}\sqrt{[AB^2-(PA-PB)^2]} + \tfrac{1}{2}(\pi+\theta)c - \tfrac{1}{2}(\pi-\theta)c\cos\phi. \tag{9}$$

At B, $\theta=0$, $A=0$, $\phi=\gamma$, $N(B) = \frac{1}{2}\pi c(1-\cos\gamma)$, (10)

$$N(P_i) - N(B) = A + \tfrac{1}{2}(\pi+\theta)c - \tfrac{1}{2}(\pi-\theta)c\cos\phi - \tfrac{1}{2}\pi c(1-\cos\gamma). \tag{11}$$

At K, $\theta=\gamma$, $\phi=0$, $A=a=c\sin\gamma$,

$$N(K) = c\sin\gamma + \tfrac{1}{2}(\pi+\gamma)c - \tfrac{1}{2}(\pi-\gamma)c = c\sin\gamma + c\gamma, \tag{12}$$

$$N(K_i) - N(B) = c\sin\gamma + c\gamma + \tfrac{1}{2}\pi c(1-\cos\gamma). \tag{13}$$

15. Denoting by σ_i the electrical density at P_i, on the inside of the bowl,

$$4\pi\sigma_i = +\frac{dV_i}{dr} = -\frac{1}{y}\frac{dN(P_i)}{cd\phi} \tag{1}$$

and the charge on the interior of the bowl swept out by the revolution of the arc P_iB is

$$\int 2\pi\sigma_i ycd\phi = \tfrac{1}{2}N(P_i) - \tfrac{1}{2}N(B), \tag{2}$$

and the total charge on the interior of the bowl is

$$\tfrac{1}{2}N(K_i) - \tfrac{1}{2}N(B) = \tfrac{1}{2}(c\sin\gamma + c\gamma) - \tfrac{1}{4}\pi c(1-\cos\gamma). \tag{3}$$

At P_0 on the exterior of the bowl the sign of A must be changed in the S. F., and ω_0, ω_i interchanged, making

$$N(P_0) = -A + c\omega_0 - c\omega_i\cos\phi$$
$$= -\tfrac{1}{2}\sqrt{[AB^2-(PA-PB)^2]} + \tfrac{1}{2}(\pi-\theta)c - \tfrac{1}{2}(\pi+\theta)c\cos\phi, \tag{4}$$

$$N(K_0) = -c\sin\gamma - c\gamma. \tag{5}$$

The electrical density σ_0 on the outside of the bowl, at P_0, is then given by

$$4\pi\sigma_0 = -\frac{dV_0}{dr} = \frac{1}{y}\frac{dN(P_0)}{cd\phi}, \tag{6}$$

and the charge on the outside of the part of the bowl swept out by P_0B is

$$\int 2\pi\sigma_0 ycd\phi = \tfrac{1}{2}N(B) - \tfrac{1}{2}N(P_0), \tag{7}$$

and the total charge on the outside is

$$\tfrac{1}{2}N(B)-\tfrac{1}{2}N(K_0)=\tfrac{1}{2}(c\gamma+c\sin\gamma)+\tfrac{1}{4}\pi c(1-\cos\gamma). \tag{8}$$

The total charge on the bowl, outside and inside, is then

$$\tfrac{1}{2}N(K_0)-\tfrac{1}{2}N(K_i)=c\gamma+c\sin\gamma \tag{9}$$

at potential π, so that the capacity of the bowl is

$$\frac{c\gamma+c\sin\gamma}{\pi}=\frac{\text{arc } AKB+\text{chord } AOB}{2\pi}=\frac{\text{meridian girth}}{2\pi}, \tag{10}$$

this is the radius of a sphere of the same girth.

This verifies for the complete sphere, and a flat circular disc AB.

The difference of the charge on the part of the bowl swept out by KP, outside and inside, is

$$\begin{aligned}\int 2\pi(\sigma_0-\sigma_i)\,yc\,d\phi &= \tfrac{1}{2}N(P_0)-\tfrac{1}{2}N(K_0)+\tfrac{1}{2}N(K_i)-\tfrac{1}{2}N(P_i)\\ &=\tfrac{1}{2}(\pi-\theta)c-\tfrac{1}{2}(\pi+\theta)c\cos\phi-\tfrac{1}{2}(\pi-\theta)c\cos\phi+\tfrac{1}{2}(\pi+\theta)c\\ &=\pi c(1-\cos\phi)=\frac{\text{surface of the part}}{2c},\end{aligned} \tag{11}$$

so that the difference $\sigma_0-\sigma_i$ is constant $=\dfrac{1}{4c}$, $\quad 4\pi(\sigma_0-\sigma_i)=\dfrac{\pi}{c}$. (12)

16. At P_i on the bowl

$$\begin{aligned}A&=\tfrac{1}{2}\sqrt{(4c^2\sin^2\gamma-16c^2\cos^2\tfrac{1}{2}\gamma\sin^2\tfrac{1}{2}\phi)}\\ &=2\cos^2\tfrac{1}{2}\gamma\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)},\end{aligned} \tag{1}$$

$$\omega_i=\tfrac{1}{2}(\pi+\theta),\quad \omega_0=\tfrac{1}{2}(\pi-\theta), \tag{2}$$

$$\frac{d\omega_i}{d\phi}=-\frac{d\omega_0}{d\phi}=\tfrac{1}{2}\frac{d\theta}{d\phi}=\frac{\tfrac{1}{2}\cos\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}, \tag{3}$$

$$\cos\tfrac{1}{2}\theta=\frac{\cos\tfrac{1}{2}\gamma}{\cos\tfrac{1}{2}\phi},\quad \sin\tfrac{1}{2}\theta=\frac{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}{\cos\tfrac{1}{2}\phi}, \tag{4}$$

$$\tan\omega_0=\cot\tfrac{1}{2}\theta=\frac{\cos\tfrac{1}{2}\gamma}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}, \tag{5}$$

$$N(P_i)=2c\cos\tfrac{1}{2}\gamma\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}+c\omega_i-c\omega_0\cos\phi, \tag{6}$$

$$\begin{aligned}\frac{dN(P_i)}{c\,d\phi}&=-\frac{\tfrac{1}{2}\cos\tfrac{1}{2}\gamma\sin\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}-\tfrac{1}{2}(1+\cos\phi)\frac{d\theta}{d\phi}+\omega_0\sin\phi\\ &=-\frac{\cos\tfrac{1}{2}\gamma\sin\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}+\omega_0\sin\phi=-(\tan\omega_0-\omega_0)\sin\phi,\end{aligned} \tag{7}$$

$$4\pi\sigma_i=-\frac{1}{c\sin\phi}\frac{dN(P_i)}{c\,d\phi}=\frac{\tan\omega_0-\omega_0}{c}, \tag{8}$$

$$4\pi(\sigma_0-\sigma_i)=\frac{\pi}{c}=\frac{\omega_i+\omega_0}{c}, \tag{9}$$

$$4\pi\sigma_0=\frac{\omega_i+\tan\omega_0}{c}=\frac{\omega_i-\tan\omega_i}{c}. \tag{10}$$

If the potential is U, instead of π, these values of σ and the charge in (9), § 15, must be multiplied by $\frac{U}{\pi}$, making

$$4\pi\sigma_i = \frac{U}{\pi c}(\tan\omega_0 - \omega_0), \quad 4\pi\sigma_0 = \frac{U}{\pi c}(\omega_i - \tan\omega_i), \tag{11}$$

$$4\pi\sigma_i = \frac{U}{\pi c}\left[\frac{AK'}{\sqrt{(AK^2 - KP_0^2)}} - \tan^{-1}\frac{AK'}{\sqrt{(AK^2 - KP_0^2)}}\right], \tag{12}$$

$$\tan\omega_0 = -\tan\omega_i = \frac{AK'}{\sqrt{(AK^2 - KP_0^2)}}. \tag{13}$$

Draw the circle, centre K' and radius $K'P_0$, cutting KA in R, R'; then,

$$AR^2 = K'R^2 - K'A^2 = K'P_0^2 - K'A^2 = KA^2 - KP_0^2;$$

$$ARK' = \omega_0, \quad AK'R = \tfrac{1}{2}\theta, \quad RK'R' = \theta. \tag{14}$$

Anywhere on the axis G on the convex side of the bowl,

$$A = 0, \quad \sin\omega = \frac{OA}{AG} = \sin CGA, \quad \sin\omega' = \frac{CG\sin\gamma}{AG} = \sin CAG,$$

$$\omega = CGA, \quad \omega' = \pi - CAG, \quad \phi = 0, \tag{15}$$

$$N(G) = A + c\omega - c\omega' = a + c(CGA + CAG)a - c\gamma = c(\sin\gamma - \gamma), \tag{16}$$

$$N(G') = A + c\omega' - c\omega = a + c\gamma = a(\sin\gamma + \gamma). \tag{17}$$

And anywhere on the axis at H on the concave side of the bowl, beyond C,

$$A = 0, \quad \omega = CHA, \quad \omega' = CAH, \quad \phi = \pi, \tag{18}$$

$$N(H) = a + c\omega + c\omega' = a + c\gamma = c(\sin\gamma + \gamma). \tag{19}$$

Thus N is $a + c\gamma$ on the concave side along the axis HK, and changes to $a - c\gamma$ along the prolongation KG, in crossing the bowl to the convex side.

17. For the hydrodynamical application to the axial motion U of the circular base AB through an infinite liquid, the velocity function (V. F.) V and stream function (S. F.) N are given by

$$V = Ux\frac{\tan\omega - \omega}{\frac{1}{2}\pi}, \quad N = \tfrac{1}{2}Uy^2\frac{\omega - \frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi} \tag{1}$$

at P; and at P',

$$V' = Ux'\frac{\tan\omega' - \omega'}{\frac{1}{2}\pi}, \quad N' = \tfrac{1}{2}Uy'^2\frac{\omega' - \frac{1}{2}\sin 2\omega'}{\frac{1}{2}\pi}; \tag{2}$$

and the combination

$$V + \frac{c}{r}V', \text{ with } N + \frac{r}{c}N', \tag{3}$$

will serve for Basset's expression of the V. F. and S. F. in the liquid motion due to the axial velocity U of the bowl, in agreement with that given in his "Hydrodynamics," I, pp. 153–156.

Fixed in the current, in an axial stream U past the circular base AB,

$$V=-Ux\left(1-\frac{\tan\omega-\omega}{\frac{1}{2}\pi}\right),\quad N=\tfrac{1}{2}Uy^2\left(1-\frac{\omega-\frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi}\right), \tag{4}$$

with $\omega=\frac{1}{2}\pi$, $N=0$ over the disc AB; while the V. F.

$$\frac{2N\cos\psi}{y}=Uy\left(1-\frac{\omega-\frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi}\right)\cos\psi \tag{5}$$

will give the potential of the electric field, uniform and of potential $Uy\cos\psi$ across the axis Ox, when the field is disturbed by the disc AB, earthed to zero potential; with similar extensions when the disc is replaced by the bowl.

If the bowl is to earth in an axial field Ux, the potential of the field is changed to

$$V=U\left(\frac{c\omega\cos\gamma}{\pi}+x\frac{\tan\omega-\omega}{\pi}\right)+U\frac{c}{r}\left(\frac{c\omega'\cos\gamma}{\pi}+x'\frac{\tan\omega'-\omega'}{\pi}\right)$$
$$-U(c\cos\gamma-x). \tag{6}$$

(Basset, I, p. 154; Gallop, Q. J. M., XXI, p. 256), with the S. F.

$$N=U\left(\frac{Ac\cos\gamma}{\pi}+\tfrac{1}{2}y^2\frac{\omega-\frac{1}{2}\sin 2\omega}{\pi}\right)$$
$$+U\frac{r}{c}\left(\frac{A'c\cos\gamma}{\pi}+\tfrac{1}{2}y'^2\frac{\omega'-\frac{1}{2}\sin 2\omega'}{\pi}\right)-\tfrac{1}{2}Uy^2. \tag{7}$$

18. Similar extensions can be made with applications of the conical angle Ω, Ω', instead of plane angles ω, ω'.

With the P. F. of the material of the bowl

$$V=\Omega+\frac{c}{r}\Omega'=\Omega+\frac{r'}{c}\Omega', \tag{1}$$

$V=\Omega+\Omega'=2\Omega$ over the spherical surface $AK'B$; but over the bowl AKB,

$$\Omega_i=2\pi+\Omega_1,\quad \Omega_0=-2\pi+\Omega_1,\quad \Omega+\Omega'=\Omega_i+\Omega_0=2\Omega_1, \tag{2}$$

so that V is not constant over the bowl AKB, or the rest of the sphere.

To obtain a constant potential over the surface, take

$$V=\Omega-\frac{c}{r}\Omega', \tag{3}$$

which is zero over $AK'B$, and 4π over AKB; and this V can serve for the electric potential when the part $AK'B$ of the sphere is to earth, and the other part AKB is changed to potential 4π, the insulation being made perfect along the line of separation of the two parts AKB, $AK'B$.

The associated S. F. can be written down, as the difference of those given in § 13,

$$N=-AQ-aP+c\Omega\cos\gamma+c\Omega'\cos\phi. \tag{4}$$

But along the insulating circle AB, $A=a$, $P+Q=\infty$; and the charge on each bowl is infinite, with an infinite electromotive force across the line of perfect insulation, so here is an electrical version of the old dynamical paradox of an irresistible force, push, or blow, applied to an immovable body.

The question was proposed for the special case of two hemispheres in the *Mathematical Tripos*, II, 1912, and the answer given in a series, but the series is divergent and the result infinite, as it should be.

19. With the potential of the bowl,

$$V=\Omega+\frac{c}{r}\Omega', \tag{1}$$

$$\frac{dV}{dr}=\frac{d\Omega}{dr}+\frac{c}{r}\frac{d\Omega'}{dr}-\frac{c}{r^2}\Omega'=-\frac{c}{r^2}\Omega', \tag{2}$$

$$\frac{dV}{rd\phi}=\frac{d\Omega}{rd\phi}+\frac{c}{r}\frac{d\Omega'}{rd\phi}=-\frac{Q}{r}, \tag{3}$$

$$\frac{dV}{dx}=-\frac{dV}{dr}\cos\phi+\frac{dV}{rd\phi}\sin\phi=\frac{c}{r^2}\Omega'\cos\phi-\frac{Q}{r}\sin\phi; \tag{4}$$

and as this is a P. F. at P, there is another P. F. at P',

$$\frac{c}{r'^2}\Omega\cos\phi-\frac{Q'}{r'}\sin\phi=\frac{r^2}{c^3}(\Omega\cos\phi-Q\sin\phi), \tag{5}$$

or another P. F. at P,

$$\frac{r}{c^2}(\Omega\cos\phi-Q\sin\phi). \tag{6}$$

The associated S. F. can be obtained from

$$\begin{aligned}\frac{dN}{dx}&=-r\sin\phi\frac{dV}{dy}=-r\sin\phi\left(\frac{dV}{dr}\sin\phi+\frac{dV}{rd\phi}\cos\phi\right)\\&=r\sin\phi\left(\frac{c}{r^2}\Omega'\sin\phi+\frac{Q}{r}\cos\phi\right)\\&=\frac{c}{r}\Omega'\sin^2\phi+Q\sin\phi\cos\phi=M_1,\end{aligned} \tag{7}$$

suppose, and there is another S. F. at P,

$$M_2=\frac{r}{c}\left(\frac{c}{r'}\Omega\sin^2\phi+Q'\sin\phi\cos\phi\right)=\frac{r^2}{c^2}(\Omega\sin^2\phi+Q\sin\phi\cos\phi). \tag{8}$$

Then
$$M_2-M_1=\left(\frac{r^2}{c^2}\Omega-\frac{c}{r}\Omega'\right)\sin^2\phi+\left(\frac{r^2}{c^2}-1\right)Q\sin\phi\cos\phi \tag{9}$$

is a S. F., zero over $AK'B$, where $r=c$, $\Omega=\Omega'$, and over the bowl AKB,

$$M_1-M_2=4\pi\sin^2\phi=4\pi\frac{y^2}{c^2} \tag{10}$$

Thus M_1-M_2 is the S. F. of the liquid motion at the instant when the spherical lid AKB is dropped on the spherical bowl $AK'B$, or lifted again, the liquid velocity being great where the windage is small.

An interpretation can also be given to the motion where the V. F. is

$$\frac{M_1-M_2}{r\sin\phi}\cos\psi=\left[\left(\frac{r}{c^2}\,\Omega-\frac{c}{r^2}\,\Omega'\right)\sin\phi+\left(\frac{r}{c^2}-\frac{1}{r}\right)Q\cos\phi\right]\cos\psi, \tag{11}$$

the lid AKB sliding across the bowl $AK'B$.

20. Figures of revolution can also be made in fig. 1 about the axis Oy. For a uniform rod AB, or a confocal prolate spheroid with an electric charge E,

$$V=\frac{2E}{AB}\,\mathrm{th}^{-1}\frac{AB}{PA+PB},\quad N=E(PA-PB), \tag{1}$$

obtained from the integral for the potential of a uniform rod

$$\begin{aligned}\int_{-a}^{a}\frac{dy'}{\sqrt{[x^2+(y'-y)^2]}}&=\mathrm{sh}^{-1}\frac{y+a}{x}-\mathrm{sh}^{-1}\frac{y-a}{x}=\mathrm{ch}^{-1}\frac{r_1}{x}-\mathrm{ch}^{-1}\frac{r_2}{x}\\&=\mathrm{sh}^{-1}\frac{a(r_1+r_2)-y(r_1-r_2)}{x^2}=\mathrm{ch}^{-1}\frac{r_1r_2-y^2+a^2}{x^2}\\&=2\,\mathrm{th}^{-1}\sqrt{\frac{r_1r_2-x^2-y^2+a^2}{r_1r_2+x^2-y^2+a^2}}=2\,\mathrm{th}^{-1}\frac{2a}{r_1+r_2}=2\,\mathrm{th}^{-1}\frac{2y}{r_1-r_2}.\end{aligned} \tag{2}$$

Then

$$V=\mathrm{sh}^{-1}\frac{y+a}{x}=\mathrm{ch}^{-1}\frac{r_1}{x},\quad N=r_1, \tag{3}$$

gives the P. F. and S. F. for a positive semi-infinite rod Ay, and negative rod Ay'. With the break at O, and $a=0$,

$$V=\mathrm{sh}^{-1}\frac{y}{x}=\mathrm{sh}^{-1}\tan\theta=\mathrm{ch}^{-1}\sec\theta=\log(\sec\theta+\tan\theta),\quad N=r, \tag{4}$$

so that the stream sheets are spherical and of uniform thickness, the flow issuing from a pole, and disappearing at the other pole, with velocity inversely as the distance from the axis.

In the conformal representation on a Mercator chart, there would be a uniform current, North and South. The S. F.

$$N=E(PA+PB)\text{ has the P. F.}\frac{2E}{AB}\,\mathrm{th}^{-1}\frac{2y}{r_1+r_2}=\frac{2E}{AB}\,\mathrm{th}^{-1}\frac{PA-PB}{AB}, \tag{5}$$

as required for confocal hyperboloids of revolution about the focal line AB, and excentricity $AB\div(PA-PB)$.

The typical element of a P. F. is $\frac{1}{r}$, as of a point source or sphere; and then the S. F. is $\cos\theta$ or $\sin\theta$ according as Ox or Oy is the axis.

But the simplest element of a S. F. is r, as in (4), for a line source Oy, and a line sink Oy'.

The single line source Oy would have the

$$\text{S. F. } N=r-y, \text{ and V. F. } V=\operatorname{sh}^{-1}\frac{y}{x}+\log x=\log(r+y), \tag{6}$$

and surfaces of constant N and V are orthogonal confocal paraboloids, with focus at O.

21. For a magnetic molecule, or sphere magnetized uniformly or moving in infinite liquid in direction Ox,

$$V=-\frac{d}{dx}\frac{1}{r}=\frac{x}{r^3}, \quad N=\frac{d}{dx}\cos\phi=-\sin\phi\frac{d\phi}{dx}=\frac{\sin^2\phi}{r}=\frac{y^2}{r^3}, \tag{1}$$

$$\frac{dN}{dx}=-y\frac{dV}{dy}=-\frac{3xy^2}{r^5}, \quad \frac{dN}{dy}=y\frac{dV}{dx}=\frac{2y}{r^3}-\frac{3xy^2}{r^5}. \tag{2}$$

For a magnetized oblate spheroid, on a focal circle AB, magnetized axially, or for the liquid motion due to axial velocity U,

$$V=-\frac{UxA(\lambda)}{2B(0)}, \qquad N=\frac{\frac{1}{2}Uy^2B(\lambda)}{B(0)}, \tag{3}$$

$$A(\lambda)=\tan\omega-\omega, \quad B(\lambda)=\tfrac{1}{2}\omega-\tfrac{1}{4}\sin 2\omega, \quad \sin\omega=\frac{AB}{PA+PB}, \tag{4}$$

and $A(0)$, $B(0)$ the value of $A(\lambda)$, $B(\lambda)$ over the surface of the spheroid.

For a prolate spheroid with a focal line AB, magnetized or moving axially,

$$V=-\frac{UyA(\lambda)}{2B(0)}, \qquad N=\frac{\frac{1}{2}Uy^2B(\lambda)}{B(0)}, \tag{5}$$

$$A(\lambda)=\tfrac{1}{4}\operatorname{sh}2\zeta-\tfrac{1}{2}\zeta, \quad B(\lambda)=\zeta-\operatorname{th}\zeta, \quad \operatorname{th}\zeta=\sin=\frac{AB}{PA+PB}=\frac{2a}{r_1+r_2}, \tag{6}$$

and then

$$B(\lambda)=\operatorname{th}^{-1}\frac{2a}{r_1+r_2}-\frac{2a}{r_1+r_2},$$

$$A(\lambda)=\tfrac{1}{4}\left(\frac{2}{\frac{r_1+r_2}{2a}-\frac{2a}{r_1+r_2}}-\operatorname{sh}^{-1}\frac{2}{\frac{r_1+r_2}{2a}-\frac{2a}{r_1+r_2}}\right) \tag{7}$$

22. The expression for the potential W of the solid homogeneous ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1 \tag{1}$$

can be derived from the components of the attraction, in the manner of §§ 8, 10,

A. J. M., p. 387, for the lens, by treating the potential as a homogeneous function of the second degree in x, y, z, and α, β, or γ, so that

$$2W = \alpha\frac{dW}{d\alpha} + x\frac{dW}{dx} + y\frac{dW}{dy} + z\frac{dW}{dz}. \tag{2}$$

It is not so difficult then (Thomson and Tait, § 494), to prove, as in Dirichlet's manner that the components of the attraction are given by

$$\frac{dW}{dx},\ \frac{dW}{dy},\ \frac{dW}{dz} = -\int_\lambda^\infty \frac{4\pi G\rho\alpha\beta\gamma\,(x,\ y,\ z)}{\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2}\,\frac{d\psi}{P(\psi)}, \tag{3}$$

$$P^2(\psi) = 4\cdot\psi+\alpha^2\cdot\psi+\beta^2\cdot\psi+\gamma^2, \tag{4}$$

and the confocal ellipsoid through the attracted point (x, y, z) is

$$\frac{x^2}{\lambda+\alpha^2} + \frac{y^2}{\lambda+\beta^2} + \frac{z^2}{\lambda+\gamma^2} = 1; \tag{5}$$

while $\alpha\dfrac{dW}{d\alpha}$, due to the uniform swelling of the ellipsoid in (1), whereby $(\alpha, \beta, \gamma, p)$ increases slightly by $k(\alpha, \beta, \gamma, p)$, p denoting the perpendicular from the centre on a tangent plane.

This makes dW the P. F. of a film, or coat of matter like paint, of thickness kp and superficial density $\sigma = k\rho p$, equivalent of an electrical film of superficial electrical density $kG\rho p$, and charge

$E = kG\rho\int p\,dS = kG\rho$ (three times the volume of the ellipsoid, or $4\pi\alpha\beta\gamma$),

and then $\sigma = \dfrac{Ep}{4\pi G\alpha\beta\gamma}$. (6)

Assuming the expression for the potential of this electrical distribution in equilibrium as

$$\int_\lambda^\infty \frac{E\,d\psi}{P(\psi)}, \tag{7}$$

this makes, with $d\alpha$ for $k\alpha$,

$$dW = \int \frac{4\pi kG\rho\alpha\beta\gamma\,d\psi}{P(\psi)},\quad \alpha\frac{dW}{d\alpha} = \int \frac{4\pi G\rho\alpha\beta\gamma\,d\psi}{P(\psi)} \tag{8}$$

Thence the complete expression of W as

$$W = \int_\lambda^\infty \left(1 - \frac{x^2}{\psi+\alpha^2} - \frac{y^2}{\psi+\beta^2} - \frac{z^2}{\psi+\gamma^2}\right)\frac{2\pi G\rho\alpha\beta\gamma\,d\psi}{P(\psi)}, \tag{9}$$

in which each term has received a physical interpretation.

If (x, y, z) is inside the ellipsoid, the lower limit λ must be replaced by zero.

23. In Professor Andrew Gray's method, "On the Attraction of a Spherical and Ellipsoidal Shell" (*Proceedings Edinburgh Mathematical*

Society, 1914), he produces CE in figure 3 to meet the concentric sphere through P in E', and then AE' is parallel to $P'E$ and equal to EP,

$$\frac{dF}{G\sigma}=\frac{dS\cos\theta}{Ep^2}=\frac{CA^2}{CP^2}\frac{dS'\cos\theta}{AE'^2}=\frac{c^2}{r^2}d\Omega', \tag{1}$$

where $d\Omega'$ is the conical angle subtended at A by dS' at E'.

He extends the method to the ellipsoidal shell through E in figure 4, drawing the confocal ellipsoid through P, and taking A, P, and E, E' corresponding points on the ellipsoids, so that $AE'=EP$.

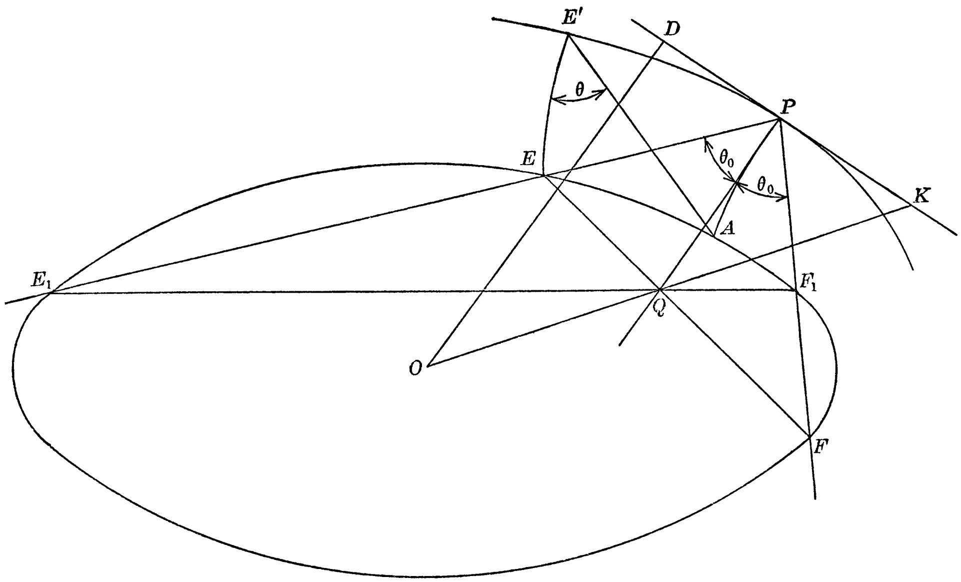

FIG. 4.

Draw the diameter parallel to AE, cutting the tangent plane at A, E in W_0, W; and denote the perpendicular from the centre O on the tangent plane at A, E by p_0, p; and denote the angle which AE makes with the normal at A, E by ϕ_0, ϕ; then $p_0\sec\phi_0=OW_0$, $p\sec\phi=OW$; and the plane through O and the line of intersection of the tangent planes at A, E bisects AE, so that

$$OW_0=OW,\quad p_0\sec\phi_0=p\sec\phi. \tag{2}$$

Similarly with ϖ_0, ϖ the perpendicular from the centre on the tangent planes at P, E', and θ_0, θ the angles which EP, AE' make with the normal at P, E', Professor Gray finds

$$\varpi_0\sec\theta_0=\varpi\sec\theta; \tag{3}$$

the equivalent of $OV_0=OV$, if OV_0, OV, parallel to EP, AE' cut the tangent planes at P, E' in V_0, V.

For if (x, y, z), (x', y', z') are the coordinates of P, E,

$$\frac{PE}{OV_0}=\frac{\text{perpendicular from } E \text{ on the tangent plane at } P}{\text{perpendicular from } O \text{ on the tangent plane at } P}$$

$$=1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}, \quad (4)$$

and this ratio is unaltered in $\frac{AE'}{OV}$, because the coordinates of the corresponding points A, E' are

$$x''=x\sqrt{\frac{\alpha^2}{\lambda+\alpha^2}}, \quad y''=y\sqrt{\frac{\beta^2}{\lambda+\beta^2}}, \quad z''=z\sqrt{\frac{\gamma^2}{\lambda+\gamma^2}}, \quad (5)$$

$$x'''=x'\sqrt{\frac{\lambda+\alpha^2}{\alpha^2}}, \quad y'''=y'\sqrt{\frac{\lambda+\beta^2}{\beta^2}}, \quad z'''=z'\sqrt{\frac{\lambda+\gamma^2}{\gamma^2}}, \quad (6)$$

so that

$$x''x'''=xx', \quad y''y'''=yy', \quad z''z'''=zz', \quad (7)$$

$$\frac{AE'}{OV}=1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}=\frac{PE}{OV_0} \quad (8)$$

But from the fundamental property of corresponding points

$$AE'^2=\left(x\sqrt{\frac{\alpha^2}{\lambda+\alpha^2}}-x'\sqrt{\frac{\lambda+\alpha^2}{\alpha^2}}\right)^2+\dots.$$

$$=x^2\frac{\alpha^2}{\lambda+\alpha^2}-2xx'+x'^2\frac{\lambda+\alpha^2}{\alpha^2}+\dots.$$

$$=x^2\left(1-\frac{\lambda}{\lambda+\alpha^2}\right)-2xx'+x'^2\left(\frac{\lambda}{\alpha^2}+1\right)+\dots.$$

$$=x^2+y^2+z^2-\lambda-2xx'-2yy'-2zz'+\lambda+x'^2+y'^2+z'^2$$

$$=(x-x')^2+(y-y')^2+(z-z')^2=PE^2. \quad (9)$$

$$OV_0=OV, \quad \varpi_0 \sec\theta_0=\varpi\sec\theta. \quad (10)$$

Draw a cone of small conical angle $d\Omega$ from the vertex at P to cut out area elements dS, dS_1 of the ellipsoid at E, E_1, the axis PEE_1 at an angle ϕ, ϕ_1 with the normal at E, E_1, so that

$$dS=PE^2\sec\phi d\Omega, \quad dS_1=PE_1^2\sec\phi_1 d\Omega. \quad (11)$$

Then with superficial density on the ellipsoid, proportioned to p the perpendicular on the tangent plane, the ratio of the attraction of these elements at E, E_1 on P is

$$\frac{\dfrac{pdS}{PE^2}}{\dfrac{p_1dS_1}{PE_1^2}}=\frac{p\sec\phi}{p_1\sec\phi_1}=1. \quad (12)$$

Thus, if P was inside the ellipsoid, the attraction of the elements would be equal and opposite, and the attraction of the shell is zero.

But with P external the normal PQ of the confocal through P is the axis of the enveloping cone; another elementary cone PF_1F can be drawn equally inclined to PQ, of the same small conical angle $d\Omega$, and contributing an equal attraction; also EF, E_1F_1 pass through Q, and make equal angles with PQ, where Q is the pole of the tangent plane at P with respect to the ellipsoid through E.

The surface of the ellipsoid can be exhausted by such pairs of cones of equal attraction, equally inclined to the normal PQ, so that the resultant attraction at P of the ellipsoidal shell is along the normal of the confocal through P, which confocal is thus a level surface.

Moreover, if the shell is divided by the plane of contact of the tangent cone from P, the two parts of the shell exert equal attraction on P.

24. With dS' the element of surface at E' on the ellipsoid PE', corresponding to dS at E on AE,

$$\frac{\varpi dS'}{pdS} = \text{ratio of corresponding conical elements of volume with vertex at } O$$
$$= \text{ratio of whole volume of ellipsoids} = \sqrt{\frac{\lambda+\alpha^2\cdot\lambda+\beta^2\cdot\lambda+\gamma^2}{\alpha^2\beta^2\gamma^2}} \tag{1}$$

and then with superficial density $k\rho p$ at E, the normal component dF of the attraction at P of the element dS at E is given by

$$\frac{dF}{kG\rho} = \frac{p\cos\theta_0 dS}{PE^2} = \frac{\varpi\cos\theta_0 dS'}{PE^2}\sqrt{\frac{\alpha\beta\gamma}{(\lambda+\alpha^2\cdot\lambda+\beta^2\cdot\lambda+\gamma^2)}}$$
$$= \frac{\varpi_0\cos\theta dS'}{AE'^2}\frac{2\alpha\beta\gamma}{P(\lambda)} = \varpi_0 d\Omega'\frac{2\alpha\beta\gamma}{P(\lambda)}. \tag{2}$$

Thus, for the whole ellipsoid, $\Omega'=4\pi$, as A is inside the ellipsoid PE', and

$$\frac{F}{kG\rho} = 4\pi\omega_0\frac{2\alpha\beta\gamma}{P(\lambda)} = 4\pi\varpi_0\frac{\text{volume of ellipsoid } AE}{\text{volume of ellipsoid } PE'}. \tag{3}$$

The potential U of this shell is then the work required to carry P off to infinity, and so

$$U = \int F d\varpi_0, \text{ with } \varpi_0 d\varpi_0 = \tfrac{1}{2}\,d\lambda. \tag{4}$$

The electric charge E, or mass of the shell, is given by

$$E = \int kG\rho p dS = 4\pi kG\rho\alpha\beta\gamma, \tag{5}$$

$$U = \int_\lambda^\infty \frac{4\pi kG\rho\alpha\beta\gamma d\lambda}{P(\lambda)} = \int_\lambda^\infty \frac{Ed\lambda}{P(\lambda)} \tag{6}$$

and the level surfaces are the confocal ellipsoids of the system

$$\frac{x^2}{\lambda+\alpha^2}+\frac{y^2}{\lambda+\beta^2}+\frac{z^2}{\lambda+\gamma^2}=1. \tag{7}$$

And conversely, U in (6) is the potential of a mass E, if its external level surfaces are given by the system in (7).

This is Professor Gray's proof; but our desideratum is to refer Ω' to some internal point P', analogous in the sphere of figure 3 to the inverse point of P.

The pole of the tangent plane at P with respect to the ellipsoid AE will be at Q on the normal PQ, with coordinates

$$\frac{\alpha^2 x}{\lambda+\alpha^2},\quad \frac{\beta^2 y}{\lambda+\beta^2},\quad \frac{\gamma^2 z}{\lambda+\gamma^2}; \tag{8}$$

and PQ will be the normal at Q of the ellipsoid through Q, homothetic with the ellipsoid through P.

If EQ makes an angle θ with the normal at E,

$$\cos\theta=\frac{px'}{a^2}\,\frac{x'-\dfrac{\alpha^2 x}{\lambda+\alpha^2}}{EQ}+\ldots, \tag{9}$$

$$EQ\cos\theta=p\left(1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}\right), \tag{10}$$

$$EP\cos\theta_0=\varpi_0\left(1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}\right), \tag{11}$$

$$\frac{EQ}{EP}=\frac{p\sec\theta}{\varpi_0\sec\theta_0}, \tag{12}$$

$$\frac{dF}{kG\rho}=\frac{p\cos\theta_0 dS}{EP^2}=\frac{EQ}{EP^3}\varpi_0\cos\theta dS=\frac{EQ^3}{EP^3}\varpi_0 d\Omega'. \tag{13}$$

But this is not a constant multiple of $d\Omega'$ on the ellipsoid, but only on the sphere, and so the analogy breaks down with P' at Q, and some other position must be found for P', say on the line of force PA.

25. In a reduction to a standard form of the elliptic integral, take

$$\alpha^2<\beta^2<\gamma^2,\ \text{and then}\ s_1>s_2>s_3, \tag{1}$$

on putting

$$\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2=m^2(s-s_1,\ s-s_2,\ s-s_3), \tag{2}$$

$$\frac{d\psi}{P(\psi)}=\frac{ds}{m\sqrt{S}},\quad \int_\lambda^\infty\frac{\sqrt{(\gamma^2-\alpha^2)}\,d\psi}{P(\psi)}=\int_\sigma^\infty\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}=eK, \tag{3}$$

$$\mathrm{sn}^2 eK=\frac{s_1-s_3}{\sigma-s_3}=\frac{\gamma^2-\alpha^2}{\lambda+\gamma^2},\quad \mathrm{cn}^2 eK=\frac{\lambda+\alpha^2}{\lambda+\gamma^2},\quad \mathrm{dn}^2 eK=\frac{\lambda+\beta^2}{\lambda+\gamma^2} \tag{4}$$

$$\kappa^2=\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2},\quad \kappa'^2=\frac{\beta^2-\alpha^2}{\gamma^2-\alpha^2}. \tag{5}$$

Then, putting $\lambda=0$, $E=1$ in (6), § 24, the reciprocal of the capacity of the ellipsoid (1), § 22, is

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\frac{eK}{\sqrt{(\gamma^2-\alpha^2)}}=\frac{\mathrm{cn}^{-1}\frac{\alpha}{\gamma}}{\sqrt{(\gamma^2-\alpha^2)}}. \tag{6}$$

Thus, to determine λ for an ellipsoid of double capacity, we take the formulas for $\frac{1}{2}eK$,

$$\mathrm{sn}^2\tfrac{1}{2}eK=\frac{1-\mathrm{cn}eK}{1+\mathrm{dn}eK},\quad \frac{\gamma^2-\alpha^2}{\lambda+\gamma^2}=\frac{\gamma-\alpha}{\gamma+\beta}, \tag{7}$$

$$\lambda+\gamma^2=\gamma+\alpha\cdot\gamma+\beta,\quad \lambda+\beta^2=\beta+\gamma\cdot\beta+\alpha,\quad \lambda+\alpha^2=\alpha+\beta\cdot\alpha+\gamma, \tag{8}$$

(Hargreave's *Messenger of Mathematics*, August, 1912).

For an oblate ellipsoid, $\beta=\gamma=\alpha\sec\theta$, $\varkappa=0$,

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\int\frac{d\psi}{2(\psi+\beta^2)\sqrt{(\psi+\alpha^2)}}=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\cos^{-1}\frac{\alpha}{\gamma}$$
$$=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\sin^{-1}\frac{\sqrt{(\gamma^2-\alpha^2)}}{\gamma}=\frac{\theta}{\gamma\sin\theta}=\frac{\theta}{\alpha\tan\theta}. \tag{9}$$

For a prolate ellipsoid, $\alpha=\beta$, $\varkappa=1$,

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\int\frac{d\psi}{2(\psi+\alpha^2)\sqrt{(\psi+\gamma^2)}}=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\mathrm{ch}^{-1}\frac{\gamma}{\alpha}$$
$$=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\mathrm{sh}^{-1}\frac{\sqrt{(\gamma^2-\alpha^2)}}{\alpha}; \tag{10}$$

giving the capacity of the sphere by the radius, when $\alpha=\beta=\gamma$.

For an elliptic plate $\alpha=0$, $\mathrm{cn}^{-1}0=K$, $\varkappa'=\frac{\beta}{\gamma}$, making the capacity $\gamma\div K$; agreeing with the circular plate when $\beta=\gamma$, $\varkappa=0$, $K=\frac{1}{2}\pi$. Or, as in § 8, the capacity of the elliptic plate is the Gauss A. G. M. of β and γ, divided by $\frac{1}{2}\pi$.

To determine λ for an ellipsoid of n-fold capacity of the elliptic plate requires the elliptic functions of K/n.

Thus, as before in (8), for double capacity, $\lambda=\beta\gamma$ with $\alpha=0$.

For three-fold capacity, from the formula

$$\mathrm{sn}\tfrac{1}{3}K+\mathrm{cn}\tfrac{2}{3}K=1,\quad \mathrm{sn}\tfrac{1}{3}K+\frac{\varkappa'\,\mathrm{sn}\frac{1}{3}K}{\mathrm{dn}\frac{1}{3}K}=1, \tag{11}$$

$$\frac{\sqrt{(\lambda+\gamma^2)}}{\gamma}+\frac{\beta\sqrt{(\lambda+\gamma^2)}}{\gamma\sqrt{(\lambda+\beta^2)}}=1,\quad \frac{\gamma}{\sqrt{(\lambda+\gamma^2)}}-\frac{\beta}{\sqrt{(\lambda+\beta^2)}}=1, \tag{12}$$

leading on rationalisation to

$$\lambda^4-6\beta^2\gamma^2\lambda^2-4(\beta^2+\gamma^2)\beta^2\gamma^2\lambda-3\beta^4\gamma^4=0, \tag{13}$$

a Jacobian quartic for λ.

Or, for a condenser of capacity 50 per cent greater than the plate,

$$\frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{dn}\frac{2}{3}K}+\operatorname{cn}\tfrac{2}{3}K=1,\ \sqrt{\frac{\lambda}{\lambda+\beta^2}}+\sqrt{\frac{\lambda}{\lambda+\gamma^2}}=1,\ \frac{1}{\sqrt{(\lambda+\beta^2)}}+\frac{1}{\sqrt{(\lambda+\gamma^2)}}=\frac{1}{\sqrt{\lambda}},\tag{14}$$

$$3\lambda^4+4(\beta^2+\gamma^2)\lambda^3+6\beta^2\gamma^2\lambda^2-\beta^4\gamma^4=0,\tag{15}$$

the same Jacobian quartic (13) as before, with the substitution $\left(\lambda, \frac{\beta\gamma}{\lambda}\right)$.

26. With (9), § 22, for the homogeneous ellipsoid of mass M,

$$\frac{W}{\frac{3}{2}GM}=D(\lambda)-x^2A(\lambda)-y^2B(\lambda)-z^2C(\lambda),\tag{1}$$

changing in the interior to

$$\frac{W}{\frac{3}{2}GM}=D(0)-x^2A(0)-y^2B(0)-z^2C(0),\tag{2}$$

$$D(\lambda)=\int_\lambda^\infty\frac{d\psi}{P(\psi)}=\frac{eK}{\sqrt{(\gamma^2-\alpha^2)}},\quad \operatorname{sn}^2eK=\frac{\gamma^2-\alpha^2}{\lambda+\gamma^2},\quad \kappa^2=\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2},\tag{3}$$

$$A(\lambda),\ B(\lambda),\ C(\lambda)=\int_\lambda^\infty\frac{1}{\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2}\frac{d\psi}{P(\psi)}=-\frac{2dD(\lambda)}{d(\alpha^2,\beta^2,\gamma^2)},\tag{4}$$

$$A(\lambda)+B(\lambda)+C(\lambda)=\frac{2}{P(\lambda)},\tag{5}$$

and then with

$$\frac{1}{\psi+\alpha^2},\ \frac{1}{\psi+\beta^2},\ \frac{1}{\psi+\gamma^2}=\frac{1}{\gamma^2-\alpha^2}\left(\frac{\operatorname{sn}^2u}{\operatorname{cn}^2u},\ \frac{\operatorname{sn}^2u}{\operatorname{dn}^2u},\ \operatorname{sn}^2u\right)\tag{6}$$

$$A(\lambda),\ B(\lambda),\ C(\lambda)=\frac{1}{(\gamma^2-\alpha^2)^{\frac{3}{2}}}\int_0^{eK}\left(\frac{\operatorname{sn}^2u}{\operatorname{cn}^2u},\ \frac{\operatorname{sn}^2u}{\operatorname{dn}^2u},\ \operatorname{sn}^2u\right)du,\tag{7}$$

three Elliptic Integrals of the Second Kind,

$$(\gamma^2-\alpha^2)^{\frac{3}{2}}[A(\lambda),\ B(\lambda),\ C(\lambda)]$$
$$=\frac{\operatorname{zs}(1-e)K-eE}{\kappa'^2},\ \frac{e(E-\kappa'^2K)-\operatorname{zn}(1-e)K}{\kappa^2\kappa'^2},\ \frac{e(K-E)-\operatorname{zn}eK}{\kappa^2}.\tag{8}$$

27. An ocean film, covering the surface of a Jacobian rotating ellipsoid, would have the depth inversely as the local gravity, or directly as p the perpendicular on the tangent plane, neglecting its self-gravitation; and so it would resemble the electrical coating or charge on the insulated ellipsoid.

But on Maclaurin's theorem, if the matter of the ellipsoid, § 22, was cut out and condensed on the surface, this film would have the same exterior potential as the solid ellipsoid if the thickness was inversely as p, as if a thin focoid of uniform density.

Maclaurin's theorem is equivalent then to saying that the external potential of the solid ellipsoid is unaltered if a confocal cavity is excavated, and the stuff condensed uniformly in the remaining focoidal shell.

Denoting by l, m, n, the direction cosines of the perpendicular p and p_1 on parallel tangent planes of the confocal ellipsoids defined by λ and λ_1,

$$p^2=(\lambda+\alpha^2)l^2+(\lambda+\beta^2)m^2+(\lambda+\gamma^2)n^2=\lambda+\alpha^2l^2+\beta^2m^2+\gamma^2n^2,$$
$$p_1^2=\lambda_1+\alpha^2l^2+\beta^2m^2+\gamma^2n^2; \tag{1}$$

and defining the thickness of the focoidal shell by the distance between the two parallel tangent planes

$$p_1^2-p^2=\lambda_1-\lambda,\quad p_1-p=\frac{\lambda_1-\lambda}{p_1+p}; \tag{2}$$

reducing for a thin shell to a thickness

$$dp=\frac{d\lambda}{2p}. \tag{3}$$

In Thomson and Tait's "Natural Philosophy" (T and T', § 494 d,) the problem is considered in the reverse way by taking a P. F.

$$V=\tfrac{1}{2}\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right),\text{ inside the ellipsoid; } V=0 \text{ in exterior space;} \tag{4}$$

so that V is continuous, but its variation is discontinuous in crossing the ellipsoid; thence the appropriate density ρ is determined by Laplace's equation,

$$4\pi G\rho=-\nabla^2V=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2},\text{ inside the ellipsoid, but outside} = 0; \tag{5}$$

and, in crossing the surface, the superficial density σ is given by

$$4\pi G\sigma=\frac{dV}{dp}=\frac{dV}{dx}\frac{px}{\alpha^2}+\ldots.=-\frac{px^2}{\alpha^4}\ldots.=-\frac{1}{p}. \tag{6}$$

Thus the focoidal film or shell of superficial density $\sigma=\dfrac{1}{4\pi Gp}$ will have the same exterior potential as the solid ellipsoid of density

$$\rho=\frac{1}{4\pi G}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right),$$

and the same mass; since, integrating over the surface S, and volume V,

$$\int\frac{dS}{4\pi Gp}=\frac{1}{4\pi G}\int\frac{ldydz+mdzdx+ndxdy}{p}$$
$$=\frac{1}{4\pi G}\int\left(\frac{xdydz}{\alpha^2}+\frac{ydzdx}{\beta^2}+\frac{zdxdy}{\gamma^2}\right)$$
$$=\frac{V}{4\pi G}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)=V\rho=M. \tag{7}$$

Thence for the focoid shell

$$\text{outside,}\ W=\tfrac{3}{2}GM\int_\lambda^\infty\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\rho^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{d\psi}{P(\psi)} \tag{8}$$

$$\text{inside,}\ W=\tfrac{3}{2}GM\int_0^\infty\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\rho^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{d\psi}{P(\psi)}$$
$$-\tfrac{3}{2}GM\frac{\alpha\beta\gamma}{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right). \tag{9}$$

So that if the solid ellipsoid (1), § 22, is not homogeneous, but stratified in confocals, the exterior potential is unaltered.

But if the strata are similar homothetic ellipsoids, then

$$W=2\pi G\alpha\beta\gamma\int_{\lambda,\,0}^\infty f\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\beta^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{d\psi}{P(\psi)}, \tag{10}$$

where the density is given by

$$\rho=f'\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right). \tag{11}$$

Thus, if there is a cavity of semi-axes $k(\alpha,\beta,\gamma)$ in a thick homeoid, the potential in the interior is given by

$$W=2\pi G\alpha\beta\gamma[f(1-k^2)-f(0)]\int_0^\infty\frac{d\psi}{P(\psi)}. \tag{12}$$

Consult memoirs on this subject by Ferrers and Dyson, in the *Quarterly Journal of Mathematics*, XIV, p. 1, 1877, and XXV, p. 259, 1890.

28. We resume here the interpretation of the terms in the potential W in (9), § 22, as we think it would strike a Maxwell or Clifford, in an investigation of the physical meaning of them, to show reason why these terms depending on the matter inside the ellipsoid and on the surface should arise in an integration extending through exterior space, beyond the attracted point away to infinity.

In his "Biographical Introduction" to Clifford's "Lectures and Essays," Sir Frederick Pollock cites his reminiscence of a walk with Clifford at Cambridge, "when Clifford explained to him in words the inner meaning of Ivory's theorem, and its geometrical conditions, omitting all the formidable apparatus of coordinates and integrals, where the chain of symbolic proof seemed artificial and dead, and failed to satisfy the reason where it compelled the understanding."

Clifford's line of argument is probably the same as that incorporated later in T and T', § 532, depending on Ivory's theorem of corresponding points.

In an associated problem of hydrodynamics or induced magnetism, we can take the function $\frac{dW}{dx}$ as a new P. F., and give a physical interpretation.

According to Maxwell (E. and M. II, § 437), $\frac{dW}{dx}$ for any attracting solid will give the magnetic potential or velocity function of the body, only however, when $\frac{dW}{dx}$ is a linear function of the coordinates x, y, z, within the body, and W is then a quadratic function in the interior.

The only case with which we are acquainted in which W is a quadratic function of the coordinates within the body is that in which the body is bounded by a complete surface of the second degree.

But in the case of a lens, bounded by portions of a spherical surface, W is not a quadratic function, so that $\frac{dW}{dx}$ will not give a velocity function for the movement of the surface, or a magnetic potential for uniform magnetization in that direction.

29. Begin by considering the hydrodynamical interpretation of the component $\frac{dW}{dx}$ of the attraction of the ellipsoid, by taking an equivalent velocity function

$$\phi = xA(\lambda), \text{ or more generally, } x[A(\lambda) - A(\lambda_1)] = \int_\lambda^{\lambda_1} \frac{x}{\psi+\alpha^2} \frac{d\psi}{P(\psi)}. \quad (1)$$

The up-gradient of ϕ in the direction of the normal of the confocal λ is then, writing A, A_1 for $A(\lambda)$, $A(\lambda_1)$,

$$\frac{d\phi}{dp} = \frac{dx}{dp}(A - A_1) + x\frac{dA}{dp} = l(A - A_1) + 2px\frac{dA}{d\lambda}$$

$$= l\left[A - A_1 + 2(\lambda+\alpha^2)\frac{dA}{d\lambda}\right] = 2l\sqrt{(\lambda+\alpha^2)}\frac{d}{d\lambda}[\sqrt{(\lambda+\alpha^2)}(A - A_1)], \quad (2)$$

or with

$$\frac{dA}{d\lambda} = -\frac{1}{\lambda+\alpha^2}\frac{1}{P(\lambda)}, \quad 2(\lambda+\alpha^2)\frac{dA}{d\lambda} = -\frac{2}{P(\lambda)} = -A - B - C, \quad (3)$$

$$\frac{d\phi}{dp} = -l(A_1 + B + C) = -lu, \quad u = A_1 + B + C, \quad (4)$$

taking the down-gradient of ϕ as the velocity; and this shows that any confocal λ may be supposed to swim in the liquid for a moment, without distortion, with this velocity u parallel to Ox.

At infinity, B and $C=0$, and $u = A_1$, so that $A_1 = 0$, $\lambda_1 = \infty$ is required to make the velocity there zero.

But in (4) the normal velocity is zero over the ellipsoid λ_2, where

$$B(\lambda_2)+C(\lambda_2)=-A(\lambda_1), \tag{5}$$

requiring $\lambda_1+\alpha^2$ to be negative, and making in (1),

$$\phi=x[A(\lambda)+B(\lambda_2)+C(\lambda_2)]. \tag{6}$$

Thus, if there is an infinite stream with velocity $-U$ parallel to Ox, and the ellipsoid λ_2 is inserted, the velocity function is changed from Ux to

$$\phi=Ux\frac{A(\lambda)+B(\lambda_2)+C(\lambda_2)}{B(\lambda_2)+C(\lambda_2)}=Ux\left(\frac{A}{B_2+C_2}+1\right); \tag{7}$$

and with the stream reduced to rest, and the ellipsoid λ_2 advancing with velocity U,

$$\phi=Ux\frac{A}{B_2+C_2}. \tag{8}$$

And generally, in the space between the ellipsoids λ_3, λ_4,

$$\phi=x\frac{u_3(A+B_4+C_4)-u_4(A+B_3+C_3)}{B_3+C_3-B_4-C_4}, \tag{9}$$

with the ellipsoid λ_3, λ_4 advancing with velocity u_3, u_4; reducing with $u_3=u_4=U$ to $\phi=-Ux$.

30. To show how the continuity of the liquid requires that $A(\lambda)$ should have the form given in (1), § 29, consider the flow across the annular section $K-K_2$ made by $x=0$ of the ellipsoid λ, and an interior ellipsoid λ_2, moving with velocity u and u_2 along Ox.

Then $uK-u_2K_2$ is the rate of increase of volume between the two half ellipsoids, and this must be made good by filling up of the flow of the liquid across the annulus, with velocity $-A+A_1$, since $x=0$ makes $\dfrac{d\phi}{dx}=A-A_1$.

The integral equation of continuity is then

$$uK-u_2K_2+\int_{\lambda_2}^{\lambda}(A-A_1)\,dK=0, \tag{1}$$

and differentiating with respect to λ for the differential equation of continuity

$$\frac{d}{d\lambda}uK+(A-A_1)\frac{dK}{d\lambda}=0,\quad \frac{d}{d\lambda}(u+A-A_1)K-\frac{dA}{d\lambda}K=0. \tag{2}$$

With the value of u from (2), (3), (4), § 29,

$$u+A-A_1=-2(\lambda+\alpha^2)\frac{dA}{d\lambda}, \tag{3}$$

$$2\frac{d}{d\lambda}\left[(\lambda+\alpha^2)\frac{dA}{d\lambda}K\right]+\frac{dA}{d\lambda}K=0,\quad 2(\lambda+\alpha^2)\frac{d}{d\lambda}\left(\frac{dA}{d\lambda}K\right)+3\frac{dA}{d\lambda}K=0, \tag{4}$$

and integrating
$$\frac{dA}{d\lambda}K(\lambda+\alpha^2)^{\frac{3}{2}}=\text{constant};\tag{5}$$

so that with
$$K=\pi\sqrt{(\lambda+\beta^2\cdot\lambda+\gamma^2)}=\frac{\frac{1}{2}\pi P}{\sqrt{(\lambda+\alpha^2)}},\tag{6}$$
$$\frac{dA}{d\lambda}=\frac{\text{constant}}{(\lambda+\alpha^2)P(\lambda)},\tag{7}$$
as in (1), § 29.

31. When the ellipsoid of § 22 is of revolution, make the circle on AB the focal circle for an oblate spheroid by putting $\beta=\gamma$, as in (9), § 25, but for a prolate spheroid take AB as the focal line, and put $\beta=\alpha$, as in (10), § 25. Then in parallel columns:

Spheroid	Oblate, on axis Ox	Prolate, on axis Oy
$\lambda+\alpha^2$	$a^2\cot^2\omega=B^2$	$\dfrac{a^2}{\text{sh}^2\zeta}=B^2$
$\lambda+\beta^2$	$a^2\sec^2\omega$	$a^2\dfrac{\text{ch}^2\zeta}{\text{sh}^2\zeta}$
$\beta^2-\alpha^2$	a^2	a^2
$\dfrac{AB}{PA+PB}$	$\sin\omega$	$\text{th}\,\zeta$
$P(\lambda)$	$2a^3\dfrac{\cos\omega}{\sin^3\omega}$	$2a^3\dfrac{\text{ch}\,\zeta}{\text{sh}^3\zeta}$
$aD(\lambda)=\int_\lambda^\infty\dfrac{a\,d\psi}{P(\psi)}$	ω	ζ
$a^3A(\lambda)$	$\tan\omega-\omega$	$\frac{1}{4}\text{sh}2\zeta-\frac{1}{2}\zeta$
$a^3B(\lambda)$	$\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega$	$\zeta-\text{th}\,\zeta$
$\dfrac{aW}{\frac{3}{2}GM}$	$\omega-\dfrac{x^2}{a^2}(\tan\omega-\omega)$ $-\dfrac{y^2+z^2}{a^2}(\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega)$	$\zeta-\dfrac{x^2+z^2}{a^2}(\frac{1}{4}\text{sh}2\zeta-\frac{1}{2}\zeta)$ $-\dfrac{y^2}{a^2}(\zeta-\text{th}\,\zeta)$
At infinity	$\omega=\sin\omega=\dfrac{a}{r}$	$\zeta=\text{th}\,\zeta=\dfrac{a}{r}$
$a^3A(\infty)=a^3B(\infty)$	$\frac{1}{3}\omega^3$	$\frac{1}{3}\zeta^3$
$\dfrac{rW(\infty)}{\frac{3}{2}GM}$	$\frac{2}{3}$	$\frac{2}{3}$
$\dfrac{a^3N}{\frac{3}{2}GM}$	$xy^2(\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega)+\frac{2}{3}x^3\tan^3\omega$	$x^2y(\frac{1}{4}\text{sh}^2\zeta-\frac{1}{2}\zeta)+\frac{2}{3}y^3\text{sh}^3\zeta$

The Hypergeometric Function in its Physical Applications.

32. Many of the functions required in a physical problem can be classed in a special case of the Hypergeometric (H. G.) Function, defined in Forsyth's "Differential Equations," Chapter VI.

Beginning with Euler's First and Second Integral, called the Gamma and Beta Function, defined by

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx=\frac{1}{n}\int_0^\infty e^{-\sqrt[n]{z}}dz, \tag{1}$$

$$B(m,n)=\int_0^1 s^{m-1}(1-s)^{n-1}ds=2\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta, \tag{2}$$

with $s=\sin^2\theta$, $1-s=\cos^2\theta$; they are connected by the relation

$$B(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}, \tag{3}$$

so that the Gamma Function alone requires tabulation, given by Bertrand (Integral Calculus) as log Γn, and between 0 and 1 for n, since $\Gamma(n+1)=n\Gamma n$.

Proceeding with a generalization, the definite integral

$$F(\alpha,\beta,\gamma,x)=A\int_0^1 s^{\beta-1}(1-s)^{\gamma-\beta-1}(1-xs)^{-\alpha}ds \tag{4}$$

defines the Hypergeometric Function, where A is chosen so as to make the function unity when $x=0$, and the function reduces to a Beta Function; so that

$$A\int_0^1 s^{\beta-1}(1-s)^{\gamma-\beta-1}ds=1,\quad \frac{1}{A}=B(\beta,\gamma-\beta). \tag{5}$$

Introducing the Elliptic Function, and putting

$$s=\text{sn}^2v,\quad 1-s=\text{cn}^2v,\quad 1-xs=\text{dn}^2v,\quad x=\varkappa^2, \tag{6}$$

$$F(\alpha,\beta,\gamma,x)=2A\int_0^K(\text{sn}v)^{2\beta-1}(\text{cn}v)^{2\gamma-2\beta-1}(\text{dn}v)^{-2\alpha+1}dv, \tag{7}$$

or with $v=eK$,

$$F(\alpha,\beta,\gamma,x)=2A\int_{e=0}^1(\text{sn}\,eK)^{2\beta-1}(\text{cn}\,eK)^{2\gamma-2\beta-1}(\text{dn}\,eK)^{-2\alpha+1}K\,de, \tag{8}$$

where β and $\gamma-\beta$ must be positive; and this is in a form analogous to Euler's Second Integral in (2).

Then F is a solution of the hypergeometric differential equation (Forsyth's D. E., Chapter VI),

$$x(1-x)\frac{d^2F}{dx^2}+[\gamma-(\alpha+\beta+1)x]\frac{dF}{dx}-\alpha\beta F=0. \tag{9}$$

Schwartz has shown that $F(\alpha,\beta,\gamma,x)$ is an algebraical function of the fourth element x in the special cases identified by Klein in his "Ikosahedron" as the polyhedral functions.

The XXIV transformations of the H. G. function in (8) are then obtained from the four arguments:

$$v+0, \quad K, \quad K+K'i, \quad K'i, \tag{10}$$

with Abel's six linear transformations for the modulus:

$$x, \quad 1-x, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x}, \quad \frac{1}{x}, \tag{11}$$

a succession of the complementary and reciprocal modulus.

33. In the special case of $\alpha+\beta=1$, $\gamma=1$, then (9), § 32, becomes the D. E. of the zonal harmonic of order n, where

$$n(n+1)=-\alpha\beta=-N, \text{ suppose, } n+\tfrac{1}{2}=\sqrt{(\tfrac{1}{4}-N)}, \tag{1}$$

and for any order n, integral, fractional, or complex, the zonal harmonic of $x=\sin^2\frac{1}{2}\theta$ is

$$P_n=F=2A\int_0^K\left(\frac{\operatorname{sn}v\operatorname{dn}v}{\operatorname{cn}v}\right)^{\sqrt{(1-4N)}=2n+1}dv=A\int_0^{2K}\left(\frac{1-\operatorname{cn}w}{1+\operatorname{cn}w}\right)^{\sqrt{(\frac{1}{4}-N)}}dw, \tag{2}$$

with $w=2v$, and a modular angle $\frac{1}{2}\theta$.

Should $\frac{1}{4}-N$ be negative, a Mehler function arises of order $p=\sqrt{(N-\frac{1}{4})}$, given by

$$F=\int_0^{2K}\cos\log\left(\frac{1-\operatorname{cn}w}{1+\operatorname{cn}w}\right)^p dw, \tag{3}$$

satisfying the D. E.

$$x(1-x)\frac{d^2F}{dx^2}+(1-2x)\frac{dF}{dx}-(p^2+\tfrac{1}{4})F=0. \tag{4}$$

If we put

$$\frac{1+\operatorname{cn}w}{1-\operatorname{cn}w}=e^{\alpha}, \quad \operatorname{cn}w=\operatorname{th}\tfrac{1}{2}\alpha, \quad \operatorname{sn}w=\operatorname{sech}\tfrac{1}{2}\alpha, \quad \operatorname{dn}w=\operatorname{sech}\tfrac{1}{2}\alpha\sqrt{(\operatorname{ch}^2\tfrac{1}{2}\alpha-x)}, \tag{5}$$

$$-\operatorname{sn}w\operatorname{dn}w\,dw=\tfrac{1}{2}\operatorname{sech}^2\tfrac{1}{2}\alpha\,d\alpha, \quad dw=\frac{-\frac{1}{2}d\alpha}{\sqrt{(\operatorname{ch}^2\frac{1}{2}\alpha-x)}}, \tag{6}$$

then with $p=\sqrt{(\frac{1}{4}-N)}$, as in (2),

$$F=A\int_{-\infty}^{\infty}e^{p\alpha}\frac{\frac{1}{2}d\alpha}{\sqrt{(\operatorname{ch}^2\frac{1}{2}\alpha-x)}}=A\int_{0,\,-\infty}^{\infty,\,0}\frac{e^{p\alpha}d\alpha}{\sqrt{(\operatorname{ch}^2\frac{1}{2}\alpha-x)}}=A\int_0^{\infty}\frac{\operatorname{ch}p\alpha\,d\alpha}{\sqrt{(\operatorname{ch}^2\frac{1}{2}\alpha-x)}}, \tag{7}$$

changing in (3), where $p=\sqrt{(N-\frac{1}{4})}$, to

$$F=A\int_0^{\pi}\frac{\cos p\alpha\,d\alpha}{\sqrt{(\operatorname{ch}^2\frac{1}{2}\alpha-x)}}. \tag{8}$$

Hobson's expression (*Phil. Trans.*, 1896) for the Mehler function of order p, solution of (4); and so a H. G. function of imaginary order, with

$$\alpha=\tfrac{1}{2}+pi, \quad \beta=\tfrac{1}{2}-pi, \quad \gamma=1, \quad x=\sin^2\tfrac{1}{2}\theta. \tag{9}$$

The Tesseral Harmonic (Hobson, *Phil. Trans.*, 1896),

$$P_{n,m}(\mu) = \frac{1}{\Pi(m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}} F\{-n,\ n+1,\ 1+m,\ \tfrac{1}{2}(1-\mu)\} \tag{10}$$

can then be expressed by the definite integral

$$\int_0^K \left(\frac{\operatorname{sn} v \operatorname{dn} v}{\operatorname{cn} v}\right)^{2n+1} (\operatorname{cn} v)^{2m}\, dv, \tag{11}$$

or

$$\int_0^{2K} \left(\frac{1-\operatorname{cn} w}{1+\operatorname{cn} w}\right)^{n+\frac{1}{2}} \left(\frac{\operatorname{dn} w+\operatorname{cn} w}{1+\operatorname{dn} w}\right)^m dw. \tag{12}$$

34. Comparing the expansion, in ascending powers of $\frac{r}{a}$ or $\frac{a}{r}$, of the typical terms $\frac{1}{R}$ and R of the P. F. and S. F., where

$$R^2 = a^2 - 2ar\mu + r^2, \tag{1}$$

$$\frac{1}{R} = \Sigma\left(\frac{r^n}{a^{n+1}} \text{ or } \frac{a^n}{r^{n+1}}\right) P_n(\mu), \tag{2}$$

$$R = \Sigma\left(\frac{r^{n+1}}{a^n} \text{ or } \frac{a^{n+1}}{r^n}\right) I_n(\mu), \tag{3}$$

then $P_n(\mu)$ is the zonal harmonic, and $I_n(\mu)$ the associated function for the S. F., and then in (2), § 4, and (2), § 6, with

$$x = \tfrac{1}{2}(1-\mu) = \sin^2 \tfrac{1}{2}\theta, \quad x' = \tfrac{1}{2}(1+\mu) = \cos^2 \tfrac{1}{2}\theta,$$

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP_n}{d\mu} + n(n+1)P_n = 0, \quad \frac{d}{dx}\left(xx'\frac{dP_n}{dx}\right) + n(n+1)P_n = 0, \tag{4}$$

$$(1-\mu^2)\frac{d^2 I_n}{d\mu^2} + n(n+1)I_n = 0, \quad xx'\frac{d^2 I_n}{dx^2} + n(n+1)I_n = 0, \tag{5}$$

so that

$$\frac{dI_n}{d\mu} = P_n, \quad I_n = \int P_n d\mu = -\frac{1-\mu^2}{n(n+1)}\frac{dP_n}{d\mu} = \frac{1}{n(n+1)}\sin\theta\frac{dP_n}{d\theta}. \tag{6}$$

And when n is an integer,

$$P_n = \frac{1}{n!}\frac{d^n(xx')^n}{dx'^n}, \quad I_n = \frac{2}{n!}\left(\frac{d}{dx'}\right)^{n-1}(xx')^n, \tag{7}$$

$$(2n+1)I_n = P_{n+1} - P_{n-1}, \tag{8}$$

$$I_n = -P_{n+1} + 2\mu P_n - P_{n-1}, \tag{9}$$

$$n(n+1)I_n = -(1-\mu^2)\frac{dP_n}{d\mu} = (n+1)(P_{n+1} - \mu P_n) = n(\mu P_n - P_{n-1}), \tag{10}$$

and so on, as discussed by Sampson on the "Stokes Current Function," *Phil. Trans.*, 1890, where the expressions given for I_n by the H. G. formulas can be replaced by definite integrals of the elliptic function.

The relations between a P. F. V and its S. F. N

$$r\sin\theta\frac{dV}{dr}=\frac{dN}{rd\theta},\quad r\sin\theta\frac{dV}{rd\theta}=-\frac{dN}{dr} \tag{11}$$

are satisfied then by typical terms, such as

$$(Ar^n+Br^{-n-1})P_n(\mu)\text{ in }V,\text{ and }[nAr^{n+1}+(n+1)Br^{-n}]I_n(\mu)\text{ in }N; \tag{12}$$

and constant V and N represent orthogonal surfaces; and I_n may be replaced by $P_{n+1}-P_{n-1}$ in (8).

35. The Hicks toroidal function (*Phil. Trans.*, 1881–4) is of similar nature, defined by

$$P_m(u)=\int_0^\pi\frac{d\theta}{(\operatorname{ch}u+\operatorname{sh}u\cos\theta)^{m+\frac12}}, \tag{1}$$

solution of the D. E.

$$\frac{d^2P}{du^2}+\coth u\frac{dP}{du}-(m^2-\tfrac14)P=0; \tag{2}$$

or writing C and S for ch u and sh u, and putting

$$C+1=2\operatorname{ch}^2\tfrac12u=2x,\quad C-1=2\operatorname{sh}^2\tfrac12u=2(x-1), \tag{3}$$

$$\frac{d^2P}{du^2}+\coth u\frac{dP}{du}=\frac{d^2P}{du^2}+\frac{C}{S}\frac{dP}{du}=\frac1S\frac{d}{du}\left(S\frac{dP}{du}\right)$$

$$=\frac{d}{dC}(C^2-1)\frac{dP}{dC}=\frac{d}{dx}x(x-1)\frac{dP}{dx}$$

$$=-x(1-x)\frac{d^2P}{dx^2}-(1-2x)\frac{dP}{dx}=(m^2-\tfrac14)P, \tag{4}$$

the H. G. D. E. with

$$\alpha+\beta=1,\quad\gamma=1,\quad\alpha\beta=-m^2+\tfrac14,\quad\alpha,\beta=\tfrac12\pm m. \tag{5}$$

Then

$$C+S\cos\theta=C+S-2S\sin^2\tfrac12\theta=e^u\Delta^2\tfrac12\theta=\frac{\operatorname{dn}^2v}{\gamma'},$$

$$\tfrac12\theta=\operatorname{am}v=\operatorname{am}eGe^{-u}=\gamma',\quad C=\tfrac12\left(\frac{1}{\gamma'}+\gamma'\right),\quad S=\tfrac12\left(\frac{1}{\gamma'}-\gamma'\right), \tag{6}$$

$$P_m(u)=\int_0^{\frac12\pi}\frac{2d\frac12\theta}{\left(\frac{\Delta\frac12\theta}{\sqrt{\gamma'}}\right)^{2m+1}}=2\sqrt{\gamma'}\int_0^G\frac{dv}{\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}}=2\sqrt{\gamma'}\int_0^G\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}dv, \tag{7}$$

as the value is unaltered on changing v into $G-v$.

The sequence equation,

$$(2m+1)P_{m+1}-4mCP_m+(2m-1)P_{m-1}=0, \tag{8}$$

a difference equation with solution P_m, is then obtained by an integration between 0 and G of the differentiation

$$\frac{d}{dv}\frac{\gamma^2}{\gamma'}\operatorname{sn}v\operatorname{cn}v\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m-1}=(2m+1)\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m+2}$$
$$-4mC\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}+(2m-1)\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m-2}. \tag{9}$$

Starting with

$$P_0=2\sqrt{\gamma'}F(\gamma),\ P_1=\frac{2E(\gamma)}{\sqrt{\gamma'}},\ \text{and then}\ P_2=\tfrac{2}{3}\left(\frac{1}{\gamma'}+\gamma'\right)\frac{2E(\gamma)}{\sqrt{\gamma'}}-\tfrac{2}{3}\sqrt{\gamma'}F(\gamma), \tag{10}$$

the sequence equation (8) will determine P_m for integral values of m.

36. In a quadric transformation of the toroidal function in (7), § 35, put

$$\varkappa=\operatorname{th}\tfrac{1}{2}u=\frac{1-\gamma'}{1+\gamma'},\quad C=\operatorname{ch}u=\frac{1+\varkappa^2}{1-\varkappa^2},\quad S=\frac{2\varkappa}{1-\varkappa^2}, \tag{1}$$

$$C+S\cos\theta=\frac{1+\varkappa\sin\psi}{1-\varkappa\sin\psi},\quad \cos\theta=\frac{\sin\psi-\varkappa}{1-\varkappa\sin\psi},$$
$$\sin\theta=\frac{\varkappa'\cos\psi}{1-\varkappa\sin\psi},\quad \tan^2\tfrac{1}{2}\theta=\frac{1+\varkappa}{1-\varkappa}\cdot\frac{1-\sin\psi}{1+\sin\psi}, \tag{2}$$

$$\frac{-d\theta}{\sqrt{(C+S\cos\theta)}}=\frac{\varkappa' d\psi}{\Delta(\psi,\varkappa)}=\varkappa' dw,\quad \psi=\operatorname{am}(w,\varkappa),\quad w=(1-2c)K, \tag{3}$$

$$P_m(u)=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\frac{1-\varkappa\sin\psi}{1+\varkappa\sin\psi}\right)^m\frac{\varkappa' d\psi}{\Delta\psi}=\int_{-L}^{L}\left(\frac{1-\varkappa\operatorname{sn}w}{1+\varkappa\operatorname{sn}w}\right)^m\varkappa' dw, \tag{4}$$

$$P_0=2\varkappa' F(\varkappa),\quad P_1=\frac{2E(\varkappa)-\varkappa'^2F(\varkappa)}{\varkappa'},\ \ldots, \tag{5}$$

$$\gamma'\operatorname{tn}^2 eG=\frac{1-\operatorname{sn}(1-2e)K}{1+\operatorname{sn}(1-2e)K},\quad \frac{\operatorname{dn}^2 eG}{\gamma'}=\frac{1+\varkappa\operatorname{sn}(1-2e)K}{1-\varkappa\operatorname{sn}(1-2e)K}. \tag{6}$$

This quadric transformation is illustrated geometrically on an ellipse of excentricity $\varkappa=\operatorname{th}\tfrac{1}{2}u$, in the connection between θ the focal angle or true anomaly from perihelion, and ψ the minor excentric angle or anomaly; and then with $v=eG$, $w=(1-2e)K$, $\theta=2\operatorname{am}eG$, $\psi=\operatorname{am}(1-2e)K$.

It is also shown on figure 5, in the expression of the potential of an anchor ring or torus, discussed by Dyson, *Phil. Trans.*, 1893.

37. In the reduction of the toroidal of the second kind

$$Q_m(u)=\int_0^\infty\frac{d\theta'}{(C+S\operatorname{ch}\theta')^{m+\frac{1}{2}}}, \tag{1}$$

substitute

$$\operatorname{ch}\tfrac{1}{2}\theta' = \sec\phi, \quad \operatorname{sh}\tfrac{1}{2}\theta' = \tan\phi, \quad d\theta' = 2\sec\phi d\phi, \tag{2}$$

$$C+S\operatorname{ch}\theta' = \frac{\Delta^2\phi}{\gamma'\cos^2\phi}, \text{ to modulus } \gamma' = e^{-u}, \tag{3}$$

$$\frac{d\theta'}{\sqrt{(C+S\operatorname{ch}\theta')}} = \frac{2\sqrt{\gamma'}d\phi}{\Delta\phi} = 2\sqrt{\gamma'}dv', \quad \phi = \operatorname{am}(v', \gamma') = \operatorname{am} fG', \tag{4}$$

$$Q_m(u) = \int_0^{\frac{1}{2}\pi}\left(\frac{\sqrt{\gamma'}\cos\phi}{\Delta\phi}\right)^{2m}\frac{2\sqrt{\gamma'}d\phi}{\Delta\phi}$$

$$= 2\gamma'^{m+\frac{1}{2}}\int_0^{G'}[\operatorname{sn}(G-v')]^{2m}dv' = 2\gamma'^{m+\frac{1}{2}}\int_0^{G'}(\operatorname{sn} v')^{2m}dv'. \tag{5}$$

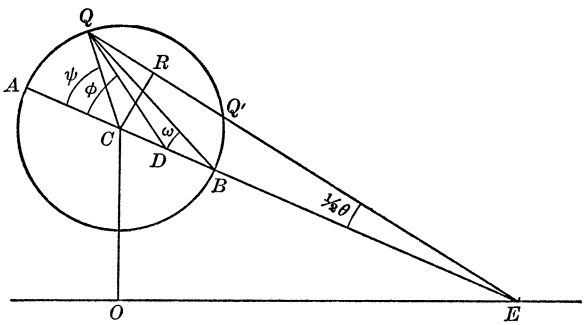

FIG. 5.

And with this toroidal in the form (this v not the same as in § 35, but employed with u, in the stereographic coordinates of A. J. M., § 24, p. 411), *i. e.* of the former volume.

$$Q_m(u) = \int_0^{\pi}\frac{\cos mv dv}{\sqrt{(2\cdot C+\cos v)}}, \tag{6}$$

$$\frac{dv}{\sqrt{(2\cdot C+\cos v)}} = \frac{\frac{1}{2}dv}{\sqrt{(\operatorname{ch}^2\frac{1}{2}u - \sin^2\frac{1}{2}v)}} = \frac{\varkappa' d\frac{1}{2}v}{\Delta(\frac{1}{2}v, \varkappa')} = \varkappa' dw',$$

$$\varkappa = \operatorname{sech}\tfrac{1}{2}u, \quad \varkappa = \operatorname{th}\tfrac{1}{2}u = \frac{1-\gamma'}{1+\gamma'}, \tag{7}$$

the complementary quadric transformation, with $\frac{1}{2}v = \operatorname{am}(w', \varkappa') = \operatorname{am} fK'$,

$$Q_m(u) = \int_0^{K'}\cos(2m\operatorname{am} w')dw'. \tag{8}$$

The connection between the reduction of (1) and (6) is made through

$$\cos\tfrac{1}{2}v = \frac{\sqrt{(C+S\operatorname{ch} v')}}{\operatorname{sh}\frac{1}{2}u + \operatorname{ch}\frac{1}{2}u\operatorname{ch} v'}, \quad \sin\tfrac{1}{2}v = \frac{\operatorname{ch}\frac{1}{2}u\operatorname{sh} v'}{\operatorname{sh}\frac{1}{2}u + \operatorname{ch}\frac{1}{2}u\operatorname{ch} v'}, \tag{9}$$

$$\frac{dv}{\sqrt{(2\cdot C+\cos v)}} = \frac{dv'}{\sqrt{(C+S\operatorname{ch} v')}}, \tag{10}$$

$$\operatorname{cn} fK' = \frac{\operatorname{cn} fG'\operatorname{dn} fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \quad \operatorname{sn} fK' = \frac{(1+\gamma')\operatorname{sn} fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \quad \operatorname{dn} fG' = \frac{1-\gamma'\operatorname{sn}^2 fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \tag{11}$$

in the complementary quadric transformation to § 36.

Then Basset's function L_m ("Hydrodynamics" I, p. 107) defined, with $c=e^{-u}=\gamma'$, by

$$mL_m=\int_0^\pi \frac{\cos(m-1)v-\cos(m+1)v}{\sqrt{(2\cdot C+\cos v)}}dv = Q_{m-1}-Q_{m+1}, \tag{12}$$

$$(2m+3)L_{m+1}-4mCL_m+(2m-3)L_{m-1}=0. \tag{13}$$

A comparison can be made too with the expressions given in Todhunter's "Functions of Laplace, Lamé, and Bessel," Chapter IV.

38. For the potential V of a solid anchor ring of mass M at a point E on the axis (Dyson, *Phil. Trans.*, 1893; Routh, "Statics," II, p. 96), we take their expression:

$$\frac{V}{M}=\frac{2}{\pi}\int_0^\pi \frac{\sin^2\psi d\psi}{EQ}, \tag{1}$$

where on figure 5, $EC=A$, $CQ=a$, $OC=c$, $ACQ=\psi=2\omega$,

$$EQ^2=r^2=A^2+2aA\cos\psi+a^2=(A+a)^2\cos^2\omega+(A-a)^2\sin^2\omega, \tag{2}$$

and with $EA=A+a=r_1$, $EB=A-a=r_2$,

$$\frac{V}{M}=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin^2 2\omega d\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}. \tag{3}$$

With r for variable, and writing R for $r_1^2-r^2\cdot r^2-r_2^2$,

$$\sin^2\omega=\frac{r_1^2-r^2}{r_1^2-r_2^2},\quad \cos^2\omega=\frac{r^2-r_2^2}{r_1^2-r_2^2},\quad \frac{d\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}=\frac{dr}{\sqrt{R}}, \tag{4}$$

$$\frac{V}{M}=\frac{16}{\pi(r_1^2-r_2^2)^2}\int_{r_2}^{r_1}\sqrt{R}dr. \tag{5}$$

Integrating by parts,

$$\int\sqrt{R}dr=r\sqrt{R}-\int[(r_1^2+r_2^2)r^2-2r^4]\frac{dr}{\sqrt{R}}$$

$$=r\sqrt{R}-2\int\sqrt{R}dr+(r_1^2+r_2^2)\int\frac{r^2dr}{\sqrt{R}}-2r_1^2r_2^2\int\frac{dr}{\sqrt{R}}$$

$$=\tfrac{1}{3}r\sqrt{R}+\tfrac{1}{3}(r_1^2+r_2^2)\int\frac{r^2dr}{\sqrt{R}}-\tfrac{2}{3}r_1^2r_2^2\int\frac{dr}{\sqrt{R}}. \tag{6}$$

Between the limits r_2 and r_1,

$$\sqrt{R}=0,\quad \int\frac{r^2dr}{\sqrt{R}}=r_1\int\Delta\omega d\omega=r_1E(\gamma),\quad \int\frac{dr}{\sqrt{R}}=\frac{F(\gamma)}{r_1},\quad \gamma'=\frac{r_2}{r_1}, \tag{7}$$

$$\frac{V}{M}=\frac{16}{3\pi r_1}\frac{(r_1^2+r_2^2)r_1^2E(\gamma)-2r_1^2r_2^2F(\gamma)}{(r_1^2-r_2^2)^2}=\frac{16}{3\pi r_1}\frac{(1+\gamma'^2)E(\gamma)-2\gamma'^2F(\gamma)}{\gamma^4}. \tag{8}$$

Or with a quadric transformation, as in Routh, "Statics," II, p. 96, with D, E inverse points in the circle AQB, and putting, as in pendulum motion,

$$CEQ=CQD=\tfrac{1}{2}\theta,\quad CDQ=CQE=\phi=\psi-\tfrac{1}{2}\theta,$$

$$\frac{V}{M}=\frac{2}{\pi A^2}\int\cos^2\phi\sqrt{(A^2-a^2\sin^2\phi)}\,d\phi=\frac{4}{\pi A}\int_0^{\frac{1}{2}\pi}\cos^2\phi\Delta(\phi,\,\varkappa)\,d\phi,$$

$$\varkappa=\frac{a}{A}=\frac{1-\gamma'}{1+\gamma'},\tag{9}$$

$$\frac{V}{M}=\frac{4}{\pi A}\int_{e=0}^{1}\operatorname{cn}^2 eK\operatorname{dn}^2 eK\,deK=\frac{4}{\pi A}\int\left(\frac{\operatorname{dn}^4 eK}{\varkappa^2}-\frac{\varkappa'^2\operatorname{dn}^2 eK}{\varkappa^2}\right)deK$$

$$=\frac{4}{\pi A}\int_0^K\left[\frac{\varkappa'^2}{\varkappa^2}\left(\frac{\operatorname{dn} v}{\sqrt{\varkappa'}}\right)^4-\frac{\varkappa'^3}{\varkappa^2}\left(\frac{\operatorname{dn} v}{\sqrt{\varkappa'}}\right)^2\right]dv=\frac{4}{\pi A}\,\frac{1}{2\sqrt{\varkappa'}}\left(\frac{\varkappa'^2}{\varkappa^2}P_2-\frac{\varkappa'^3}{\varkappa^2}P_1\right),\tag{10}$$

in the sequence equation (8), § 35, making

$$\frac{V}{M}=\frac{4}{3\pi A}\,\frac{2(1+\varkappa'^2)E(\varkappa)-\varkappa'^3F(\varkappa)}{\varkappa^2},\tag{11}$$

a quadric transformation of the expression in (8).

So also for U, the skin potential of the anchor ring, if R is the midpoint of QQ',

$$\frac{U}{M}=\int\frac{PR}{\pi A^2}\,d\phi=\frac{1}{\pi A^2}\int_0^{\pi}\sqrt{(A^2-a^2\sin^2\phi)}\,d\phi=\frac{2}{\pi A}\int_0^{\frac{1}{2}\pi}\Delta\phi d\phi=\frac{2E(\varkappa)}{\pi A}.\tag{12}$$

39. As shown on figure 1 and in § 37,

$$e^{-u}=\gamma'=\frac{PB}{PA}=\frac{EB}{EA}=\frac{1-\varkappa}{1+\varkappa},\quad \varkappa=\frac{OE}{OA}=\operatorname{th}\tfrac{1}{2}u,$$

$$\tfrac{1}{2}v=DEp=DPB=\operatorname{am} fK',\quad DEP=\operatorname{am}(1-f)K',$$

$$EBP=\operatorname{am} 2fG',\quad DBP=\operatorname{am} 2(1-f)G';$$

$$\Omega=2\pi(1-f)-4K\operatorname{zn} fK'=2\pi(1-f)-2G\operatorname{zn} 2fG'-2G\gamma'\operatorname{sn} 2fG'.$$

The point P may be supposed to circulate round the circle on DE in pendulum motion, with velocity due to the level of Ox, or proportional to BP or AP.

Starting from E, where $f=0$, $\Omega=2\pi$, as P moves along the semicircle ESD, f grows from 0 to 1, and Ω diminishes from 2π to 0. As P continues to complete the circuit back to E, f grows from 1 to 2, and Ω is negative, decreasing from 0 to -2π.

Thus 4π must be added to Ω in crossing the disc AB to make a fresh start; or P moving the other way round, clockwise, 4π must be subtracted in passing through E, as twelve hours is subtracted in passing through XII o'clock.

The angle $\phi=\operatorname{am} fG'$ in § 37 is constructed by bisecting the angle EBP in fig. 1 by By, crossing Dp in y, and then drawing the coaxial circle touching Dp

at y, on diameter de on DE, with limiting points A and B, and yd, ye bisect AyB externally and internally. Then, with centre at a, if dr, es are the tangents at d, e, the line rs passes through B, and crosses the circle on FB in q, at the same level as a, because Fq at right angles to Bq bisects rs.

Then with $DBp=\text{am}\,2fG'$, $DBr=\text{am}\,fG'=\phi$,

$$\gamma'=\frac{DB}{DA}=\frac{EB}{EA}=\frac{DB-EB}{DA-EA}=\frac{FB}{FE},\quad Fa=FB\sin^2\phi=FE\cdot\gamma'\,\text{sn}^2 fG';$$

and with $DEp=\frac{1}{2}v=\text{am}\,fK'$,

$$\text{dn}\,fK'=\frac{Bp}{BD}=\frac{py}{yD}=\frac{Ea}{aD}=\frac{EF-Fa}{DF+Fa}=\frac{1-\gamma'\,\text{sn}^2 fG'}{1+\gamma'\,\text{sn}^2 fG}.$$

Again, by the property of coaxial circles,

$$\frac{Dy}{DB}=\frac{py}{pB}=\frac{rd}{rB}=\sin\phi;\quad Dy=DF(1+\gamma')\,\text{sn}\,fG',$$

$$\text{sn}\,fK'=\sin\tfrac{1}{2}v=\sin Day=\frac{Dy}{Da}=\frac{(1+\gamma')\,\text{sn}\,fG'}{1+\gamma'\,\text{sn}^2 fG'};$$

and

$$ay^2=aB\cdot aA=FB\,\text{cn}^2 fG'\cdot FA\,\text{dn}^2 fG'=FE^2\,\text{cn}^2 fG'\,\text{dn}^2 fG',$$

$$\text{cn}\,fK'=\cos DAy=\frac{ay}{Da}=\frac{\text{cn}\,fG'\,\text{dn}\,fG'}{1+\gamma'\,\text{sn}^2 fG'}.$$

In a transformation with the stereographic coordinates (u, v) of the point P, where

$$y+ix=a\,\text{th}\,\tfrac{1}{2}(u+iv),\quad y,\,x=\frac{a(\text{sh}\,u,\ \sin v)}{\text{ch}\,u+\cos v},$$

the circular arc APB, orthogonal to the circle DPE, will cross AB at the curvilinear angle $ABP=v=2\,\text{am}\,fK'=DFp$, while the rectilinear angle $ABP=\text{am}\,2fG'$.

40. Shown on figure 2 and in § 36,

$$\varkappa=\frac{OE}{OB}=\frac{1-\gamma'}{1+\gamma'}=\text{th}\,\tfrac{1}{2}u,\ \tfrac{1}{2}\theta=\omega=ABQ=\text{am}\,eG,\ AQE=ABq=\text{am}\,(1-e)G,$$

$$AQq'=\text{am}\,(1+e)G;\ \text{ and } QE=AE\,\text{dn}\,eG,\ Eq=AE\,\text{dn}\,(1-e)G.$$

The point Q may be supposed to circulate round the circle on AB in pendulum motion, with velocity due to the level of F, or proportional to EQ or DQ.

When turned about Oy into planes at right angles, the P and Q circles of figure 1 and 2 are linked as a magnetic and electric circuit.

The quadric transformation of § 36 is shown by drawing the perpendicular OR, AX on Qq; and then, with $OER=\chi=\text{am}\,2eK$,

$$OR=OE\sin\chi=OQ\cdot\varkappa\,\text{sn}\,2eK,\quad QR=OQ\,\text{dn}\,2eK,$$
$$ER=OE\cos\chi=OQ\cdot\varkappa\,\text{cn}\,2eK,\quad RX=OQ\,\text{cn}\,2eK;$$

and in (6) § 36,

$$\gamma'\,\text{tn}^2\,eG=\frac{\text{tn}\,eG}{\text{tn}\,(1-e)G}=\frac{\tan ABQ}{\tan ABq}$$
$$=\frac{\tan AqQ}{\tan AQq}=\frac{QX}{Xq}=\frac{QR-RX}{QR+RX}=\frac{\text{dn}\,2eK-\text{cn}\,2eK}{\text{dn}\,2eK+\text{cn}\,2eK},$$
$$\frac{\text{dn}^2\,eG}{\gamma'}=\frac{\text{dn}\,eG}{\text{dn}\,(1-e)G}=\frac{QE}{Eq}=\frac{QR+RE}{QR-RE}=\frac{\text{dn}\,2eK+\varkappa\,\text{cn}\,2eK}{\text{dn}\,2eK-\varkappa\,\text{cn}\,2eK}.$$

But if L is taken in OE such that $EL=\varkappa'\cdot LO$, then $LR=LO\,\text{dn}\,2eK$; and if RL is produced to meet the circle on OE again in R',

$$OER'=ORR'=\text{am}\,(1-2e)K;$$

and

$$LR\sin ORL=OL\cos\chi,\quad \sin ORR'=\frac{OL}{LR}\,\text{cn}\,2eK=\frac{\text{cn}\,2eK}{\text{dn}\,2eK}=\text{sn}(1-2e)K=\sin\psi;$$

and the angle ψ of the results in § 36 is shown in figure 2 by the angle ORL.

Continuing the quadric transformation,

$$OLR=\text{am}\,4eL,\quad \lambda=\frac{1-\varkappa'}{1+\varkappa'}=\left(\frac{1-\sqrt{\gamma'}}{1+\sqrt{\gamma'}}\right)^2=\frac{CL}{CE},$$

if C is the centre of the circle on OE; and so on.

41. Maxwell obtains the expression of U, Ω, Ω' for a point Q on the axis, in a series of powers of $z=CQ$; and thence infers the series for a point P off the axis by introducing the zonal harmonic $Q_i(\phi)$ of the appropriate order i, as a factor of each term.

His method can be extended to the determination in the A. J. M. of W, W' (§ 3), $\dfrac{dV}{db}$ and V (§ 6), $\dfrac{dV}{d\gamma}$ (§ 15) and v (§ 16) for a thin lens; and an identification made of the result for a point Q on the axis, and thence generally for P off the axis.

The complicated dissection and integrations can then be avoided, required in G. W. Hill's method, although the results of his method must serve as a guide to a form, intangible and invisible otherwise.

Beginning with $\Omega(Q)$, the conical angle subtended at Q by the spherical segment on its circular base AB,

$$\frac{\Omega(Q)}{2\pi}=1-\frac{QO}{QA}=1-\frac{c\cos\gamma-z}{QA}=1-\left(\cos\gamma-\frac{z}{c}\right)\frac{c}{QA}$$

$$=1-\left(\cos\gamma-\frac{z}{c}\right)\left(1+\Sigma Q_i\frac{z^i}{c^i}\right), \tag{1}$$

writing Q_i for the zonal harmonic $Q_i(\gamma)$,

$$\frac{\Omega(Q)}{2\pi}=1-\cos\gamma+(Q_0-Q_1\cos\gamma)\frac{z}{c}+(Q_1-Q_2\cos\gamma)\frac{z^2}{c^2}$$

$$+\ldots\ldots+(Q_{i-1}-Q_i\cos\gamma)\frac{z^i}{c^i}\ldots\ldots, \tag{2}$$

and with $z=0$, $\Omega(C)=2\pi(1-\cos\gamma)$.

But at Q', inverse point of Q, and with the other aspect of the spherical surface,

$$\frac{\Omega(Q')}{2\pi}=\frac{Q'O}{Q'A}-1=\frac{z'-c\cos\gamma}{Q'A}-1$$

$$=(z'-c\cos\gamma)\left(\frac{1}{z'}+Q_1\frac{c}{z'^2}+\ldots\ldots+Q_i\frac{c^i}{z'^{i+1}}+\ldots\ldots\right)-1$$

$$=(Q_2-Q_1\cos\gamma)\frac{c^2}{z'^2}+\ldots\ldots+(Q_{i+1}-Q_i\cos\gamma)\frac{c^{i+1}}{z'^{i+1}}+\ldots\ldots, \tag{3}$$

and then in Ω' or $\Omega(P')$ at P', replace $\frac{c^{i+1}}{z'^{i+1}}$ by $\frac{c^{i+1}}{r'^{i+1}}Q_i(\phi)=\frac{r^{i+1}}{c^{i+1}}Q_i(\phi)$, so that

$$\frac{\Omega' r'}{2\pi c}=\Sigma(Q_{i+1}-Q_i\cos\gamma)\frac{r^i}{c^i}Q_i(\phi). \tag{4}$$

This makes

$$\frac{\Omega c+\Omega' r'}{2\pi}=c(1-\cos\gamma)+c\Sigma(Q_{i-1}-2Q_i\cos\gamma+Q_{i+1})\frac{r^i}{c^i}Q_i(\phi) \tag{5}$$

which agrees with Maxwell's result for $\frac{P}{2\pi}$ (E. and M., § 694), because

$$\int_\mu^1 Q_i d\mu=\frac{1-\mu^2}{i(i+1)}\frac{dQ_i}{d\mu}=\frac{\mu Q_i-Q_{i+1}}{i}=\frac{Q_{i-1}-Q_{i+1}}{2i+1}=Q_{i-1}-2\mu Q_i+Q_{i+1}, \tag{6}$$

and $\mu=\cos\gamma$. Also

$$\frac{P(Q)}{2\pi}=\frac{a}{QA}=\sin\gamma\left(1+\Sigma Q_i\frac{z^i}{c^i}\right),\quad \frac{P}{2\pi}=\sin\gamma\Sigma Q_i\frac{r^i}{c^i}Q_i(\phi). \tag{7}$$

Next for W, the P. F. of the circular plate AB (A. J. M., § 3),

$$\frac{W(Q)}{2\pi G\sigma}=\int_0^a\frac{ydy}{\sqrt{(y^2+QO^2)}}=QA-QO=\frac{QA^2}{QA}-QO$$

$$=c^2\left(1-2\frac{z}{c}\cos\gamma+\frac{z^2}{c^2}\right)\frac{1}{c}\left(Q_0+Q_1\frac{z}{c}+\ldots\ldots+Q_i\frac{z^i}{c^i}\ldots\ldots\right)$$

$$-c\cos\gamma+z, \tag{8}$$

$$\frac{W(Q)}{2\pi G\sigma c}=1-\cos\gamma+(1-\cos\gamma)\frac{z}{c}+(Q_0-2Q_1\cos\gamma+Q_2)\frac{z^2}{c^2}$$
$$+\ldots.+(Q_{i-2}-2Q_{i-1}\cos\gamma+Q_i)\frac{z^i}{c^i}\ldots., \tag{9}$$

$$\frac{W(Q')}{2\pi G\sigma}=Q'A-Q'O=\frac{Q'A^2}{Q'A}+c\cos\gamma-z'$$
$$=z'^2\left(1-2\frac{c}{z'}\cos\gamma+\frac{c^2}{z'^2}\right)\frac{1}{z'}\left(Q_0+Q_1\frac{c}{z'}+\ldots.+Q_i\frac{c^i}{z'^i}\ldots.\right)$$
$$+c\cos\gamma-z'=(Q_0-2Q_1\cos\gamma+Q_2)\frac{c^2}{z'}$$
$$+\ldots.+(Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{c^{i+2}}{z'^{i+1}}\ldots., \tag{10}$$

$$\frac{W(Q')z'}{2\pi G\sigma c^2}=\tfrac{1}{2}\sin^2\gamma+(Q_1-2Q_2\cos\gamma+Q_3)\frac{c}{z'}\ldots.$$
$$+(Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{c^i}{z'^i}\ldots. \tag{11}$$

$$\frac{W(Q)c+W(Q')z'}{2\pi G\sigma c^2}=\tfrac{1}{2}(1-\cos\gamma)(3+\cos\gamma)+\tfrac{1}{2}(1-\cos\gamma)^2(2+\cos\gamma)\frac{z}{c}\ldots.$$
$$+(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^i}\ldots., \tag{12}$$

$$\frac{Wc+W'r'}{2\pi G\sigma c^2}=\Sigma(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{r^i}{c^i}Q_i(\phi). \tag{13}$$

So far all these functions are P. F.'s, P, Ω, Ω', U, W, W'; but going back to A. J. M., § 3, for the expression of W in terms of Pa, QA, Ωb, here QA is a S. F., given we find by

$$QA=Pa-\Omega b-\frac{W}{G\sigma}=c\Sigma(Q_{i-2}-Q_{i-1}\cos\gamma)(P_{i-1}\cos\phi-P_i)\frac{r^i}{c^i}$$
$$=c\Sigma\frac{\sin^2\gamma\sin^2\phi Q'_{i-1}P'_{i-1}(\phi)}{(i-1)i}\frac{r^i}{c^i}. \tag{14}$$

42. The expression in (4), § 6, A. J. M., p. 384, of the axial component of the attraction of the plano-convex lens is the equivalent of the potential of the base AB, coated with density σ, less the potential of the spherical surface, coated at E with density $\sigma\mu$, $\mu=\cos OCE$. It will serve too for the magnetic potential of the lens, or equivalent current sheet round the portion of the spherical surface.

The thin lens of § 15 may then be considered a spherical segment, coated with density $\sigma(\mu-\cos\gamma)$.

The result of (4), § 6, shows that the coating $\sigma\mu$ would have potential $U(\sigma\mu)$, which can be written

$$\frac{U(\sigma\mu)}{G\sigma}=\frac{W}{G\sigma}-\frac{1}{G\rho}\frac{dV}{db}=\frac{Wc+W'r'}{3G\sigma c}+\frac{U(\sigma)\cos\gamma}{3G\sigma}$$
$$=\tfrac{1}{3}\left(\frac{W}{G\sigma}+\Omega c\cos\gamma\right)+\tfrac{1}{3}\left(\frac{W'}{G\sigma}+\Omega'c\cos\gamma\right)\frac{r'}{c}. \tag{1}$$

There is an opportunity for a geometrical interpretation on the lines of that given for $U(\sigma)$ in § 3, but a difficulty is to account for the factor $\frac{1}{3}$.

To verify this in Maxwell's manner at a point Q on the axis, we have to evaluate

$$\frac{U(\sigma\mu, Q)}{2\pi G\sigma}=\int_{\cos\gamma}^{1}\frac{\mu d\mu}{\sqrt{(c^2-2cz\cos\gamma+z^2)}}=\frac{1}{c}\int\left(1+\Sigma Q_i\frac{z^i}{c^i}\right)\mu d\mu;$$

requiring $\int_\mu^1 Q_i\mu d\mu$. (2)

Mr. J. R. Wilton gives me the general formula

$$\int_\mu^1 Q_iQ_jd\mu=\frac{(1-\mu^2)(Q_i'Q_j-Q_iQ_j')}{(i-j)(i+j+1)}. \tag{3}$$

But if $i=j$, we must make use of Hargreave's recurring formula (Whittaker, *Analysis*, p. 212).

$$(2i+1)Q_i^2-(2i-1)Q_{i-1}^2=\frac{d}{d\mu}[(Q_i^2+Q_{i-1}^2)-2Q_iQ_{i-1}]. \tag{4}$$

We only require here the special case of $j=1$, and then make use of the ordinary formulas,

$$iQ_{i-1}-(2i+1)Q_i\mu+(i+1)Q_{i+1}=0, \tag{5}$$

$$(2i+1)\int_\mu^1 Q_i\ d\mu=Q_{i-1}-Q_{i+1}=\frac{(1-\mu^2)Q_i'}{i(i+1)}, \tag{6}$$

$$(2i+1)\int_\mu^1 Q_i\mu d\mu=i\int Q_{i-1}d\mu+(i+1)\int Q_{i+1}d\mu; \tag{7}$$

and thence we find, after reduction,

$$\int_{\cos\gamma}^1 Q_i\mu d\mu=\tfrac{1}{3}(Q_{i-2}-Q_{i-1}\cos\gamma+2Q_i\sin^2\gamma-Q_{i+1}\cos\gamma+Q_{i+2}),$$

with $\int Q_0\mu d\mu=\frac{1}{2}\sin^2\gamma$, $\int Q_1\mu d\mu=\frac{1}{3}(1-\cos^3\gamma)$. (8)

But according to the preceding expressions in (5) and (13)

$$\begin{aligned}&\frac{W(Q)c+W(Q')z'}{2\pi G\sigma c}+\frac{U(Q)\cos\gamma}{G\sigma}\\
&=\tfrac{1}{2}c(1-\cos\gamma)(3+\cos\gamma)+\tfrac{1}{2}(1-\cos\gamma)(2+\cos\gamma+\cos^2\gamma)z\ldots\\
&\quad+(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^{i-1}}\ldots\\
&\quad+c\cos\gamma(1-\cos\gamma)+\tfrac{1}{2}z\cos\gamma\sin^2\gamma\ldots\\
&\qquad\qquad+\cos\gamma(Q_{i-1}-2Q\cos\gamma+Q_{i+1})\frac{z^i}{c^{i-1}}\ldots\\
&=\tfrac{3}{2}c\sin^2\gamma+(1-\cos^3\gamma)z\ldots\\
&\quad+(Q_{i-2}-Q_{i-1}\cos\gamma+2Q_i\sin^2\gamma-Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^{i-1}}\ldots,\end{aligned} \tag{9}$$

which adds the audit up, and so the identification is complete.

Then for v, the P. F. of the thin curved lens of § 15, treated as a spherical segment, coated with density $\sigma(\mu-\cos\gamma)$,

$$\frac{v}{G\sigma}=\frac{Wc+W'r'-2Uc\cos\gamma}{3G\sigma c},\quad \frac{v}{Gm}=\frac{Wc+W'r'-2Uc\cos\gamma}{3\pi G\sigma c^3(1-\cos\gamma)^2},\tag{10}$$

if of mass $m=2\pi\sigma c^2\int_{\cos\gamma}^{1}(\mu-\cos\gamma)\,d\mu=\pi\sigma c^2(1-\cos\gamma)^2$.

For a flat lens, with $\gamma=0$, this expression for v takes an indeterminate form, and is best evaluated independently, as in § 16, by a dissection of concentric circles and radiating straight lines.

The P. F. W of the flat disc may be evaluated at the same time in this manner, and then

$$\frac{1}{G\sigma}\frac{dW}{d\theta}=\int_0^a\frac{y\,d\theta}{PQ'}=PQ-PO-IA\cos\theta,\tag{11}$$

$$I=\int_0^a\frac{dy}{PQ'}=\mathrm{ch}^{-1}\frac{PQ}{PZ}-\mathrm{ch}^{-1}\frac{PO}{PZ}=\mathrm{sh}^{-1}\frac{c+A\cos\theta}{PZ}-\mathrm{sh}^{-1}\frac{A\cos\theta}{PZ},\tag{12}$$

$$PQ'^2=y^2+2Ay\cos\theta+A^2+b^2,\quad PQ^2=a^2+2Aa\cos\theta+A^2+b^2,$$
$$PZ^2=A^2\sin^2\theta+b^2,\quad PO^2=A^2+b^2,\tag{13}$$

$$\frac{dI}{d\theta}=-\frac{Aa\cos\theta+A^2+b^2}{PZ^2}\cdot\frac{A\sin\theta}{PQ}+\frac{PO\cdot A\sin\theta}{PZ^2}\tag{14}$$

43. The integral $\int I\,d\theta$ is intractable; it arises in the skin potential of the curved wall of a circular cylinder, and is shown graphically by a quadrature on the Mercator chart, as explained in the *Trans. American Math. Society*, October, 1907, §§ 53, 66.

But $\int IA\cos\theta\,d\theta$ is tractable, and integrating by parts

$$\int_0^{2\pi}IA\cos\theta\,d\theta=(IA\sin)_0^{2\pi}-\int\frac{dI}{d\theta}A\sin\theta\,d\theta$$
$$=\int\frac{Aa\cos\theta+A^2+b^2}{PZ^2}\cdot\frac{b^2d\theta}{PQ}-\int(Aa\cos\theta+A^2+b^2)\frac{d\theta}{PQ}$$
$$+PO\int\left(1-\frac{b^2}{PZ^2}\right)d\theta,\tag{15}$$

in which the first integral is $(2\pi-\Omega)b$, as shown in the *Trans. American Math. Society*, § 48, by a dissection of the circular area AB into sector elements $\frac{1}{2}a^2d\theta$, and then

$$\frac{W}{G\sigma}=\int PQ\,d\theta-2\pi PO+(2\pi-\Omega)b+QA-\frac{A^2+b^2}{a}\int\frac{d\theta}{PQ}+2\pi\cdot PO-2\pi b$$
$$=\frac{A^2+a^2+b^2}{a}P-2QA+(2\pi-\Omega)b+QA-\frac{A^2+b^2}{a}P-2\pi b$$
$$=Pa-QA-\Omega b,\tag{16}$$

as before.

The flat lens may be considered a disc, coated with density

$$\sigma\left(1-\frac{y^2}{a^2}\right)=\sigma\frac{AQ'\cdot Q'B}{OA^2},$$

and its potential is shown in § 16 to be expressible by the three elliptic integrals, of the I, II, and III kind, P, Q, and Ω.

A formula of integration by parts will show that a similar expression is obtained for a density $\sigma\left(1-\frac{y^2}{a^2}\right)^n$, or otherwise for a density varying as some even power of y.

But for an odd power, the intractable integral $\int I d\theta$ arises, in addition to the elliptic integrals.

A formula of reduction is obtained by the integration of the relation

$$\frac{d}{dy}(y^nPQ')=[(n+1)y^{n+1}+(2n+1)y^nA\cos\theta+ny^{n-1}(A^2+b^2)]\frac{1}{PQ'}. \quad (17)$$

In the definite integral the algebraical part vanishes at both limits.

Mr. Bromwich has expressed the results in a series in the *Proc. London Math. Society*, September, 1912.

On the electrified disc, and on a flattened oblate spheroid, the density is $\sigma\left(1-\frac{y^2}{a^2}\right)^n$, with $n=-\frac{1}{2}, \frac{1}{2}$; and the result for the potential is non-elliptic and given already. A similar formula of reduction will give the result for a density $\sigma\left(1-\frac{y^2}{a^2}\right)^{i-\frac{1}{2}}$, where i is an integer; and the evaluation should attract a careful worker, when the need arises in a physical problem.

So also for the S. F. of these coatings of superficial density, to be worked out as an exercise.

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